Using Ambient Noise Fields for Submarine Location

Team #525 for the Mathematical Contest in Modeling

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1 Restatement of the Problem

The world's oceans naturally contain a certain amount of noise that is created by such sources as surface waves, boats passing by, animals that sing or click, and storms. Our job is to determine if this noise can be used to detect a submarine, and determine its location, direction of travel, speed, and size.

The submarine is assumed not to make any noise on its own, and we are to use only measurements of the ambient noise field to locate the submarine.

2 The First Model

2.1 Assumptions

Here we list our major assumptions for the first model of a submarine detection system. Minor assumptions are given in the text as they are needed.

For our first model, we assume that the measurements of sound are to be made by an array of transducers (piezo-electric crystal microphones) suspended from a boat on the surface of the ocean. If the submarine exists, then it is, of course, somewhere beneath the boat, and not surfaced. Refer to Figure 1 for a picture that summarizes many of our assumptions.

- Most of the ambient sound is generated near the surface of the sea, and is of uniform intensity.

- Sound created from the surface of the sea propogates downward as plane waves. This requires that the surface be rather calm, Sea State 0 to 3.[1]

- The length, height and width of the submarine are very small compared to the depth of the ocean. This excludes cases where the submarine is in very shallow water.

- The length, height, and width of the submarine are large compared to the wavelength of the ambient noise, or at least the noise that we will use to detect it. (This is true for most submarines and frequencies.)

- The submarine will be no more than 500 m deep. Most submarines operate best between 100 m and 300 m, and descend to 500 m only when trying to escape detection. Submarines that go that deep risk hull damage due to the high pressure.

- Refraction of sound in water is insigificant at submarine depths, but significant below those depths. Therefore, sound travels in straight rays for our purposes, but bends before reaching the sea bottom. This topic is treated in greater detail in Appendix E.

- The submarines that will be detected with this technique are all made of standard steel or similarly reflective materials. Steel reflects almost 100% of any sound waves that contact it when it is immersed in water.

- There is no lateral variation in the attributes of sea water. That is, temperature, pressure, and density vary only with depth. This would not be true in exceptional locations, such as the boundary of the Gulf Stream. Also, the submarine and boat are located far enough from land that echoes and interference from land fronts are negligible.

- The salinity of sea water is constant 35 parts per thousand. (In reality, it varies by about 10%, but that small variation is ignored in this model.)

The assumption that most of the sound comes from the surface requires some justification. There are several main sources of noise in the ocean:

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[1]The *sea state* is a general measure of how rough the surface of the ocean is, similar to the Beaufort wind force scale. A sea state of 0 indicates a perfectly calm ocean, and 9 indicates a hurricane.

Figure 1: Picture of a boat with transducers, submarine, and planar noise waves
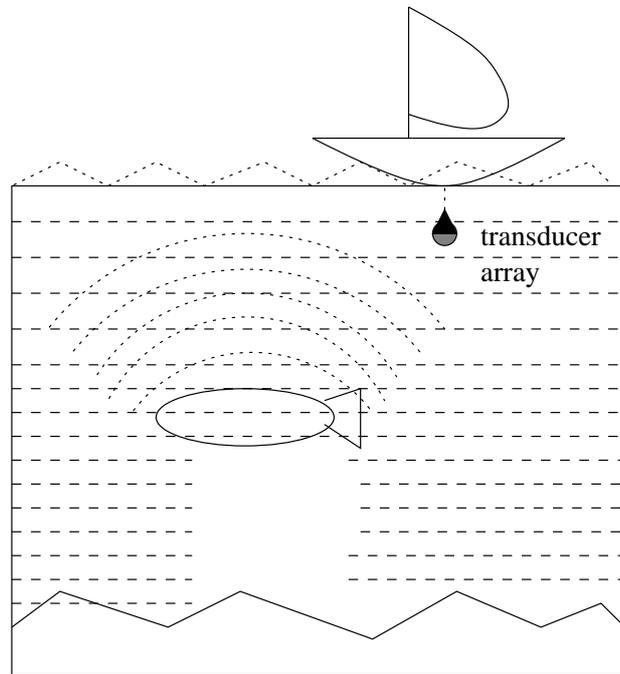
1. Thermal noise caused by motion of the ocean molecules, which is mostly high frequency, 50 kHz or more.

2. Surface noise caused by waves. This is the dominant source in deep water and has frequencies from 100 Hz to 50 kHz.

3. Noise caused by snapping shrimp, porpoises, and other aquatic creatures.

4. Artifical noise from boats and harbors, which may be the dominant source of noise below 1 kHz.

5. Sound from rain and storms.

6. Noise from waves crashing on the beach.

7. Sound caused by currents flowing over the bottom, usually of very low frequency.

8. Noise from earthquakes and volcanoes, also of very low frequency.

The surface noise includes the noise from waves, most organisms, boats, rain, and storms. Sound from harbors and crashing surf is not influential, since it was assumed that the submarine and detection equipment are out in the open sea. Noise from the bottom currents and earthquakes is low frequency, and our model will not make use of such sound.

Most of the noise in the sea is between 100 Hz and 10 kHz, no matter what the sea state, and so we select 200 Hz as the frequency to be scanned for. (The selection of 200 Hz will be made clear later in the paper.) The intensity of the sound does depend on the sea state.

Since the sound we are interested in is created at the surface, it tends to propogate downward as a planar wave, as predicted by Huygen's principle. Each point on the wave front creates a spherical secondary wave front which has the same speed and frequency as the original wave front. The front as a whole moves forward and traces out the envelope of the wavelets. Therefore, a planar wave front continues to move as a plane and a spherical wave front continues to move as a sphere, as shown in Figure 2. Since the ambient noise is created on the planar surface of the sea, it continues to move down in a planar manner.

Figure 2: Huygen's Principle for a Flat Wavefront
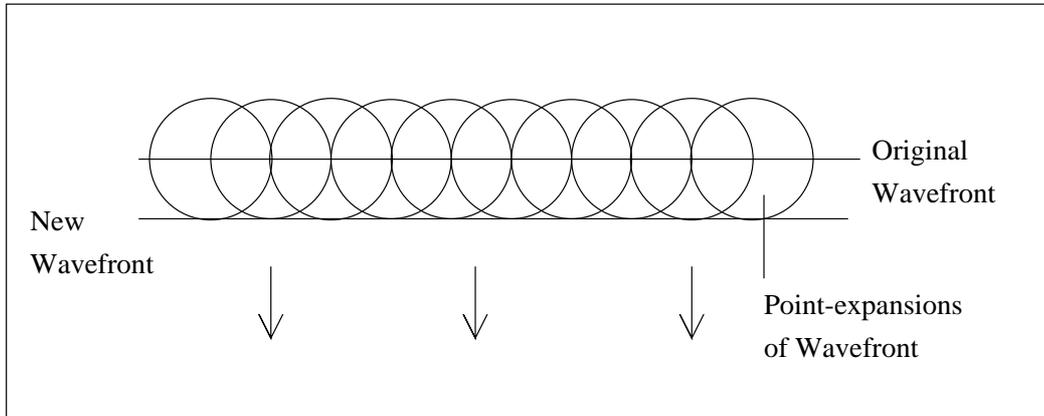
2.2   Solution to the Problem Based on the First Model

Locating the submarine requires two boats on the surface of the ocean which know each other's location. First, measurements of the background noise are made at 200 Hz for reference. Then, each boat uses an array of acoustic transducers to scan the water in every direction for sound at a frequency of 200 Hz.

When a pulse of large amplitude is detected in a particular direction, it indicates that a large object, possibly a submarine, is reflecting background noise back toward the surface. The amplitude of the pulse indicates the total of the distance to the submarine plus its depth, and the direction at which the pulse was detected indicates the direction of the submarine. The combined information from both boats is enough to pinpoint the submarine relative to the boats and determine its depth.

We recommend using change in position to determine velocity, rather than trying to use Doppler effects. Our reasoning is discussed in Section 2.5.

We also recommend using our position method to locate both ends of the vessel and triangulating to find its size. A quick summary of our methods is in Section 2.6

2.3   Computing Distance from Intensity

The transducer array will normally pick up background noise at the selected frequency of 200 Hz. If a submarine is present, some of the noise created

Figure 3: Path of a 'ray' of sound from the surface to the transducers
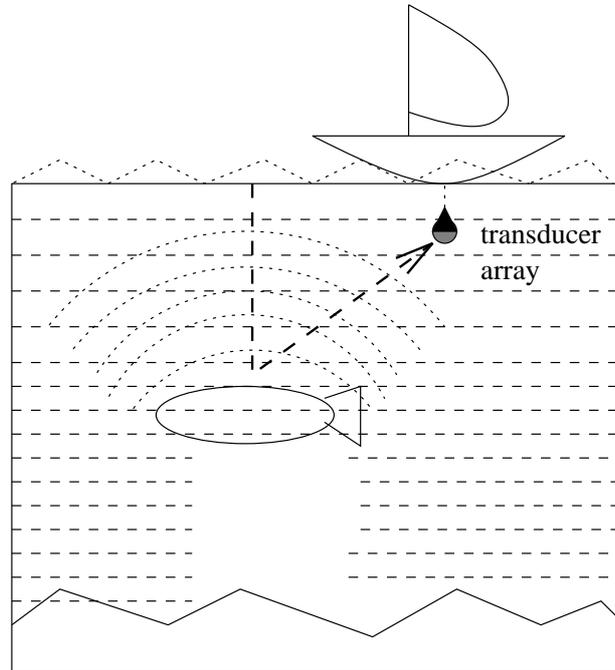
by the surface of the sea will hit it and reflect off, and some of that will be detected by the transducers.

As sound waves travel through water, they are *attenuated*, that is, they are reduced in amplitude and intensity. It is possible to determine how far a sound wave has traveled in a medium by measuring its relative attenuation.

The amplitude of the sound wave decreases with distance according to the differential equation

$$\frac{dA}{dd} = -\alpha d \quad \implies \quad A_{final} = A_{original}e^{-\alpha d} \tag{1}$$

where α is the attenuation coefficient and d is the distance the wave has traveled.

The attenuation coefficient depends on many factors, including pH, temperature, pressure, frequency of sound, and so on. However, to simplify this model, we will assume that α depends only on the frequency of the sound and that

$$\alpha = \frac{\alpha'}{1000 \, \frac{m}{km} \cdot 8.686 \, \frac{dB}{Np}} \tag{2}$$

$$\alpha' = 3.3 \times 10^{-3} + \frac{0.11f^2}{1 + f^2} + \frac{44f^2}{4100 + f^2} + 3.0 \times 10^{-4}f^2$$

where f is the frequency of the sound in kHz, α' is in dB/km, and α is in nepers/m. This equation and the conditions under which is it valid come from Reference [6].[2]

This particular formula for α' is tailored to a salinity of 35 parts per thousand, a pH of 8.0, a depth of 1000 m, and a temperature of 4°C. The depth and temperature affect the contributions of boric acid and magnesium sulfate to the absorbtion of sound in sea water. That is, they have significant effects on the value of α. These effects are ignored in this model to simplify it, but are considered in the next model.

When the transducers are placed in the water, they will detect the background noise at an amplitude of A_{bg}. If a submarine happens to come by, it will reflect the sound back toward the surface with an amplitude of A_{ref}. Since we assume the submarine to be made of steel, which reflects 98.8% of sound in underwater conditions[3], A_{ref} will be reflected at almost the same intensity as the sound waves impacting the submarine. The sound will attenuate a bit more on the way back up, and the transducers will detect an amplitude of $A_{detected} = A_{bg} + A_{final}$, where A_{final} is the final amplitude of the sound after it has travled from the surface to the submarine to the boat. (Note that we use an equation for attenuation which is based on planar waves. The reflected sound is really closer to a spherical wave, but to simplify this model, we use planar attenuation for the total distance.)

Since the sound originally has amplitude A_{bg}, we can find the distance it has travelled d.

$$\frac{A_{final}}{A_{bg}} = e^{-\alpha d}$$

from equation 1. Solving for d gives:

$$d = -\frac{\ln \frac{A_{final}}{A_{bg}}}{\alpha} \tag{3}$$

This distance is the distance traveled by the sound from the surface of the ocean, down to the submarine, and then back to the boat.

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[2] 1 neper = 1 Np = 8.686 dB. Nepers arise from using natural logarithms. Decibels arise from common logarithms.

[3] The derivation of this figure is in the appendix.

2.4   Locating the Submarine

To locate the submarine requires two boats with microphones. One boat alone does not give enough information to determine exactly where the submarine is. We will assume that the two boats know each other's relative positions, possibly through satellite positioning or some other such accurate method. (See Figure 4.) Let a be the depth of the submarine, c_1 be the distance from the submarine to the first boat, and c_2 be the distance from the submarine to the second boat. The transducer array is affixed to the boat, so we need not account for the distance from the array to the boat. Then it is possible to determine the total distances $d_1 = a + c_1$ and $d_2 = a + c_2$ from the amplitude A_{final} of the echo from the submarine.

The two boats and the submarine create a sort of underwater pyramid, of which several pieces of information are know. First, since the boats know their positions relative to each other, the distance e between them is known. The transducer arrays are able to find the direction of the signal they are detecting, which allows us to know θ_1 and θ_2, which are angles from the line connecting the two boats to a point on the surface of the ocean directly above the submarine. (See figure 5.)

From θ_1 and θ_2, it is easy to determine θ_3:

$$\theta_3 = \pi - \theta_1 - \theta_2 = 180° - \theta_1 - \theta_2 \tag{4}$$

The law of sines states that

$$\frac{e}{\sin \theta_3} = \frac{b_1}{\sin \theta_2} = \frac{b_2}{\sin \theta_1}$$

From there, b_1 and b_2 are easy to solve for:

$$b_1 = e \frac{\sin \theta_2}{\sin \theta_3} \tag{5}$$

$$b_2 = e \frac{\sin \theta_1}{\sin \theta_3} \tag{6}$$

To find the depth a of the submarine requires knowing b_1 and b_2 from above, as well as the overall distances d_1 and d_2 as detected with the transducers. (See figure 4.)

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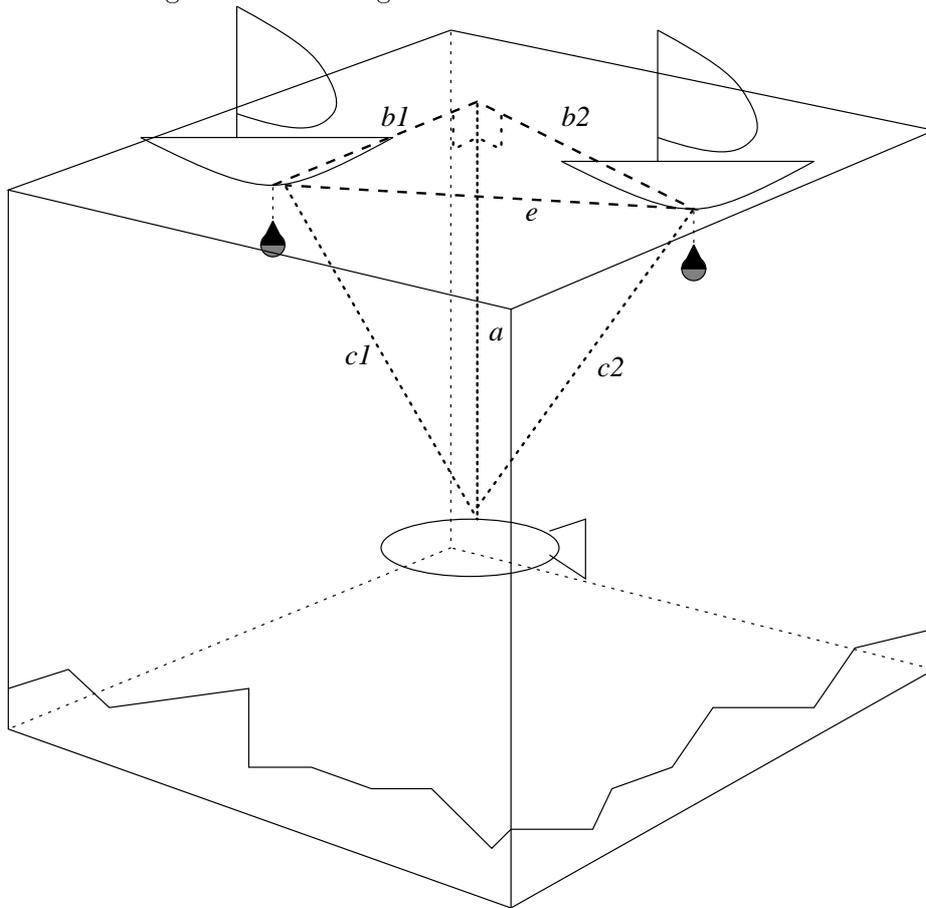
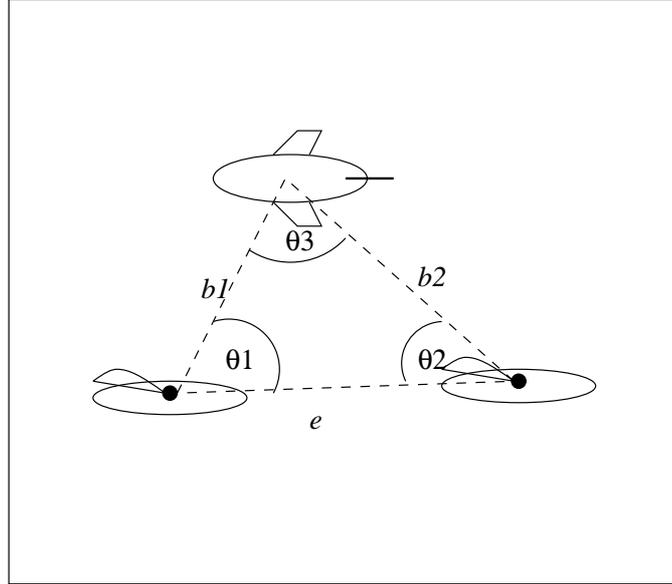Figure 4: Locating the submarine with two boats

Figure 5: Top view of boats and submarine

$$
\begin{aligned}
d_1 &= c_1 + a \\
c_1^2 &= b_1^2 + a^2 \quad \text{(from the Pythagorean theorem)} \\
d_1 &= \sqrt{b_1^2 + a^2} + a \\
d_1 - a &= \sqrt{b_1^2 + a^2} \\
d_1^2 - 2ad_1 + a^2 &= b_1^2 + a^2 \\
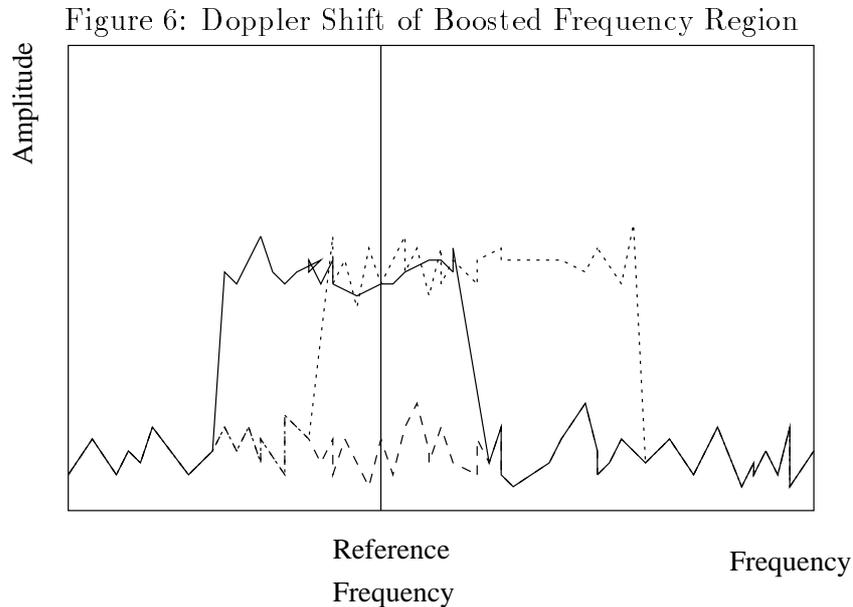d_1^2 - 2ad_1 &= b_1^2
\end{aligned}
$$

The derivation finally gives:

$$
a = \frac{d_1^2 - b_1^2}{2d_1} \tag{7}
$$

and similarly, using the other boat and triangle:

$$
a = \frac{d_2^2 - b_2^2}{2d_2} \tag{8}
$$

The values of b_1, b_2, θ_1, θ_2, and a determine the position vectors $\vec{\mathbf{p_1}}$ and $\vec{\mathbf{p_2}}$ of the submarine relative to the boats.

Figure 6: Doppler Shift of Boosted Frequency Region

2.5 Determining the Submarine's Velocity

There are two immediate strategies for determining the submarine's velocity: finding position at varying times and using a simple formula to derive velocity, or using the Doppler effect. Let us treat the latter approach first.

The Doppler effect makes the apparent frequency of a signal shift when the the signal is being reflected from a moving target. Thus, if the sound is reflected from a submarine moving away from us, the sounds we hear will seem to be of lower frequency than they were at their origins. Since we are only taking note of sounds at a particular frequency (200 Hz) in this model, monitoring shifting frequencies seems problematic at best. Thus, using Doppler shift is not our best strategy for determining velocity of the submarine. Indeed, we begin to wonder if the Doppler effect might render the submarine undetectable in a single-frequency model.

Recalling our characterization of the ambient sound frequency spectrum, however, Doppler shifts do not appear to adversely affect our ability to determine the submarine's presence. After all, the sound generated near the surface of the ocean is present in a wide range of frequencies surrounding 200 Hz, and thus the presence of a submarine will boost all of those frequencies (See Figure 6). The Doppler shift will translate the boosted region to

Figure 7: Triangulation for Determination of Size
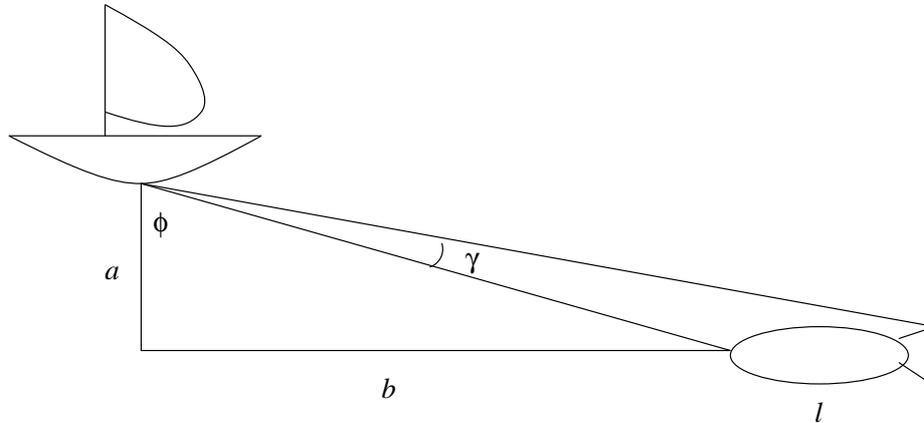
a range of higher or lower frequencies which should still include 200 Hz for submarines moving at reasonable speeds.[4]

Since the Doppler method is not practical in our one-frequency model, we turn to the simpler and more direct method of measuring position at regular intervals. The velocity of the submarine can then be determined by dividing the distance traveled by the unit of time that it took to do so. That is,

$$\vec{\mathbf{v_1}} = \frac{\Delta \vec{\mathbf{p_1}}}{\Delta t}$$

$$\vec{\mathbf{v_2}} = \frac{\Delta \vec{\mathbf{p_2}}}{\Delta t}$$

2.6   Determining the Size of the Submarine

The optimal method for determining the size of the submarine within this model is to find both of the ends of the vessel and then to triangulate.

When the transducer array detects an echo, it does not come from one exact direction, but rather from a small range of directions. Let γ be the magnitude of this angular range, as in Figure 7. The depth of the submarine is a, its length is l, and the horizontal distance from the boat to the submarine is b. Finally, ϕ is the angle from the vertical to the submarine.

We have

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[4]Reasonable speeds for modern-day combat submarines are less than 50 knots, or about 25 meters per second.

$$\tan\phi \quad = \quad \frac{b}{a}$$

$$\tan(\gamma + \phi) \quad = \quad \frac{b+l}{a}.$$

The result is

$$l = a \cdot (\tan(\gamma + \phi) - \tan(\phi)).$$

For example, a submarine of length 130 m, depth 300 m, and lateral distance 1000 m should reflect sound in an angular range of 1.8°. Thus, if the submarine is nearly under the boat and not very deep, the size should be known to a high degree of accuracy. As the submarine's lateral distance increases, however, its angular size is on the order of a degree, and accurate size determinations become much more difficult.

Another consideration is the submarine's orientation with respect to the boat. If the submarine is aligned perpendicular to the axis connecting itself with the boat, then γ must be measured as a horizontal angular size. However, if the submarine is parallel to that axis, then we need to measure a vertical angular size. Finally, if the submarine is tilted for diving or surfacing, then the calculation of its size becomes problematic. If the calculation of the vessel's size is not done correctly, it may be mistaken for a smaller submarine or even for a whale. (See Appendix F for more about whales.)

2.7   Application of the Model

Consider for example our selected frequency of 200 Hz. If $f = 0.2$ kHz, then $\alpha' = 7.973 \times 10^{-3} \frac{\text{dB}}{\text{km}}$ from equation 2, and then

$$\alpha = 9.179 \times 10^{-4} \frac{\text{Np}}{\text{km}} = 9.179 \times 10^{-7} \frac{\text{Np}}{\text{m}}$$

Suppose that measurements by the transducer array indicate an attenuation ratio of

$$\frac{A_{final}}{A_{bg}} = 0.9991$$

The total round-trip distance may be found from equation 3:

$$d = -\frac{\ln \frac{A_{final}}{A_{bg}}}{\alpha} = 981.9 \text{ m}$$

2.8   Error Analysis

The location of the submarine as given by equations 5 and 7 is not exact, but varies depending upon the accuracy of the measurements taken by the transducer array. Suppose that the transducers can measure angles to within $\pm\epsilon$ radians, that is, the angles θ_1 and θ_2 are accurate to within $\pm\epsilon$. Suppose further that the distance traveled by the sound d is somewhere between $d(1 + \delta)$ and $d(1 - \delta)$. Assume that ϵ and δ are 'small' numbers.

It is not difficult to substitute these intervals in for the measured quantities and find a maxium and minimum value for b_1, b_2, and a. The results are:

$$b_{1\,min} = \left(\frac{\sin\theta_2 - \epsilon}{\sin\theta_3 + 2\epsilon}\right) e$$

$$b_{1\,max} = \left(\frac{\sin\theta_2 + \epsilon}{\sin\theta_3 - 2\epsilon}\right) e$$

$$b_1 \in (b_{1\,min} \ldots b_{1\,max})$$

and similarly for b_2. In these equations, ϵ must be in radians.

$$a_{min} = \frac{d_1^2(1 - \delta)^2 - b_{1\,max}^2}{2d_1(1 + \delta)}$$

$$a_{max} = \frac{d_1^2(1 + \delta)^2 - b_{1\,min}^2}{2d_1(1 - \delta)}$$

$$a \in (a_{min} \ldots a_{max})$$

If, for example, the angle accuracy is $\pm 1° = 1.745 \times 10^{-2}$ rad and the measurement of d is accurate to within 5%, and the results from the transducers are $\theta_1 = 77°$, $\theta_2 = 70°$, $d_1 = 405$ m, and $e = 20$ m, then we can conclude

$$b_1 \in 31.8 \text{ m} \ldots 37.6 \text{ m}$$
$$a \in 172.4 \text{ m} \ldots 233.7 \text{ m}$$

For a derivation of these error formulas, see appendix C.

2.9   Analysis of this Model

The most serious weakness for this model is the attenuation coefficient α, particularly with the formulas in equation 2. That formula for α is intended to be used to measure the attenuation of a planar wave front, which is correct for the noise waves moving down through the water. However, the sound reflected by the submarine will be more spherical and attenuate at a different rate. This has not been taken into account.

Also, equation 2 is designed for a specific set of conditions, including pressure, temperature and so on. Those conditions are found at 1000 m of depth in the ocean. However, the standard operating range for submarines is 100 m to 300 m. Modern military submarines are also capably of descending to 500 m to avoid being detected, but cannot remain there long because the pressure can damage their hull. The submarines never go anywhere near the depth where equation 2 is accurate.

In general, a new expression of attenuation is needed, and that is the subject of our second model.

3   A Second Model with Generalized Attenuation

The value for α, the attenuation coefficient, can be generalized to include the influence of depth, temperature, frequency, and a number of other factors. We were able to locate some research which included a formula for such a generalized α. This generalized model of the attenuation coefficient is known to be accurate to within 5% of the true value for almost all oceans in the world. It is limited to frequencies of 200 Hz to 1 MHz, which is why we selected 200 Hz as the frequency to be scanned for. It is also limited to depths of up to 5000 m, which includes the entire range where submarines might be found. A summary of that work and the formula may be found in Appendix A.

According to the formula, the factor with the most influence on α is the temperature. Temperature is known to be fairly stable for the first 30-120 m of depth, then decrease steadily down to 5°C at 700 m, and then slowly decrease down to a constant 1.5°C at thousands of meters deep. Since

temperature is so dependent upon depth, we decided to create a function that approximates temperature in terms of depth. The details of the function are included in Appendix A.1. Using it, α may be expressed as a function only of depth, with other factors held constant.

It is possible to find a system of three equations with three unknowns: the depth of the submarine a, the distance from the submarine to one of the boats c, and the angle ϕ between the surface of the sea and the line from the boat to the submarine. For the derivation of the system, see Appendix A.2. This system, combined with the usual method for finding b_1 and b_2 from e and θ_1 and θ_2, again gives enough information to locate the submarine relative to the two boats.

The problem with the system of equations is that it is very difficult to solve numerically, and we were unable to solve it for an example submarine.

However, since we know that the revised α is accurate to within 5%, it is possible to explain how one might compute the error involved in locating the submarine. First, assume that α is 5% too large, and re-solve the system for a and c. Then, assume that α is 5% too small, and re-solve the system. This process gives minimum and maximum values for a and c.

4   Strengths and Weaknesses of the Model

The weaknesses of the model are:

- We ignored the effects of refraction and assumed that sound travels in straight lines, at least at the depths where submarines might be found. Refraction does tend to bend sound around significantly, and will cause our model to predict locations incorrectly. We do not know how much effect refraction has on our results.

- This model does not work near land, near the boundary of the Gulf Stream, or other places where the behavior of sound varies laterally.

- We have no numerical examples for our second model. For the first one, we could make up reasonable numbers to see how it works, but the second is computationally too difficult. We also have very little idea of how much better the second model is for that same reason.

- Our refined system of equations is impractical to solve.

- The model of temperature as a function of depth is difficult to analyze, and needs to be refined before the second model may be fully understood.

- The second model uses an equation for attenuation of a planar wave to compute the attenuation of the spherical wave reflected by the submarine. Hence, the submarine appears to be farther away than it really is.

- Scattering of sound by bubbles, plankton and other floating obstacles is not accounted for.

The strengths of our model are:

- Our model does successfully compute the location, velocity, and size of the submarine.

- It satisfies all constraints of the problem statement.

- Since our model uses measurements from two boats, it is possible to compare them and determine how reliable the computations are.

- We have expressions that tell how accurate our calculations should be given the accuracy of the transducers.

- We have provided a way to improve the accuracy by using a refined attenuation coefficient.

- The refined α is known to be accurate to within 5% in all of the world's major oceans, seas, and gulfs. Our model is therefore flexible.

A   An Expression for the Attenuation Coefficient Based on Temperature, Depth, and Frequency

There has been substantial research on the attenuation of sound in saltwater since the 1950's. For the purpose of our second model, we will use the Francois and Garrison equation for the attenuation coefficient. Their equation, which models the absorption of sound in seawater, is considered to be a sum of three factors: the boric acid contribution, the magnesium sulfate contribution, and the pure water contribution.

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 \qquad \text{in } \tfrac{\text{dB}}{\text{km}}, \text{ where}$$
$$\alpha_1 = \frac{A_1 P_1 f_1 f^2}{f^2 + f_1^2}$$
$$\alpha_2 = \frac{A_2 P_2 f_2 f^2}{f^2 + f_2^2}$$
$$\alpha_3 = A_3 P_3 f^2$$

and f is the frequency of sound, f_1 and f_2 are the relaxation frequencies of boric acid and magnesium sulfate, P_1, P_2, and P_3 are non-dimensional pressure correction factors, and A_1, A_2, and A_3 are multiplicative constants for their respective components. Note that α must be converted to Np/m before it can be used.

The variables used in solving for each components were

c the speed of sound (in units of m/s)
T the temperature (in degrees Celsius)
s the salinity (parts per thousand)
pH the acidity in water
f the frequency of the ambient noise (in units of kHz)
a the depth (in units of m)

Assigning values of 200 Hz for our frequency, 8 for our pH, 35 ppt for our salinity, we are left with the variables a and T, the depth and temperature

Figure 8: α as a function of temperature and depth (in Np/m)
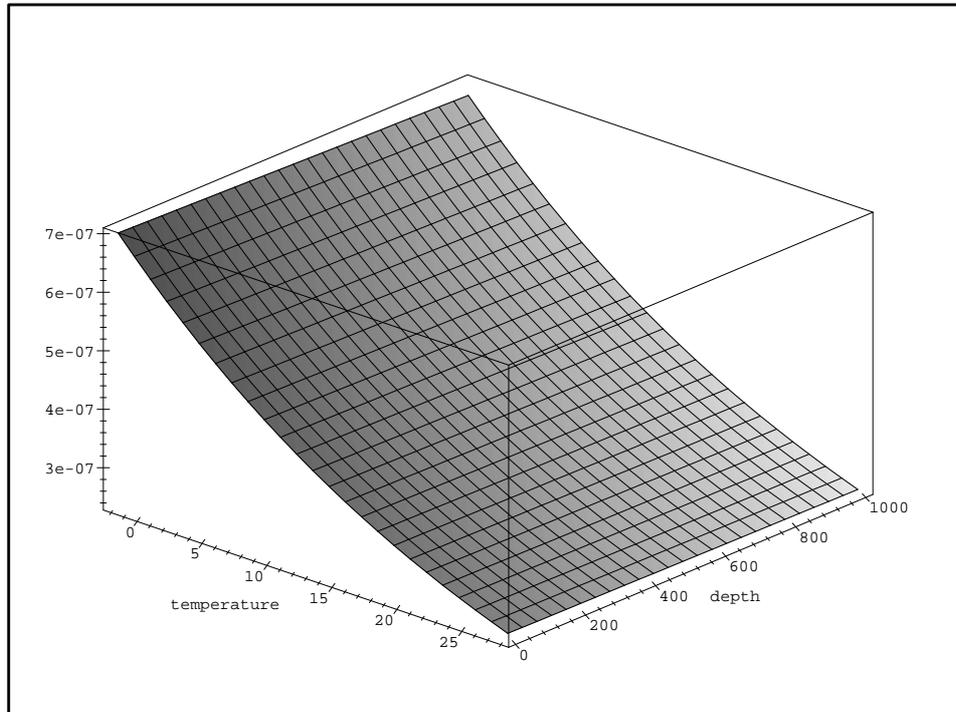
of the seawater. (The speed of sound in water is a function of depth and salinity in this model, so therefore it does not contribute a new variable to our expression.)

Ultimately, by combining α_1, α_2, α_3, which are the three components of the attenuation coefficient in terms of temperature and depth, we came up with the α function. This function is a comprehensive model of the attenuation coefficient, taking in account the boric acid, magnisium sulfate, and pure water absorption. A three dimensional graph of the attenuation function in terms of the two variables is given in Figure 8. From examining the graph closely, we noticed the depth does not have nearly as pronounced effect as temperature to the value of α.

The Francois and Garrison equation predicts absorption in natural seawater in the frequency range of 200 Hz to 1 MHz and to depths of 5000 m. The

Figure 9: Temperature as a function of depth
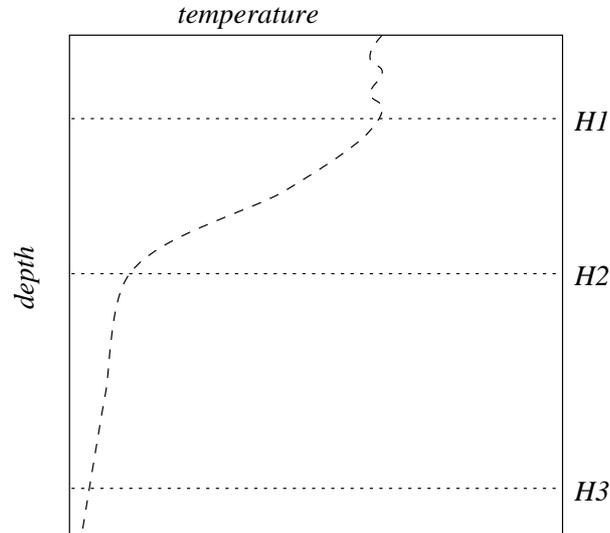
addition of temperature and depth as variables are two main improvements of this model to our first model of α. Another improvement is that the generalized equation is fitted from data collected over the Pacific and Atlantic Oceans, the Mediterranean and Red Seas, and Gulf of Aden and thus applies to all oceanic conditions with frequencies from 200 Hz to 1 MHz.

A.1 Modeling the Temperature in the Ocean

The generalized attenuation coefficient α is a function of frequency, temperature, and depth. The formula is valid for a range of frequencies including 200 Hz, which is why we selected 200 Hz as our frequency to scan for. That remains constant throughout all the following calculations.

  To generalize our method of determining distance to the submarine, we must express α as a function of depth alone, or rather, express temperature as a function of depth. To do this, we used data from the University of Hawaii's *Ocean Atlas of Hawaii*.[5]

  The temperature near the surface is roughly constant down to a depth H_1 of between 30 and 120 m, depending on the season and other factors. From

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[5] Available on the World Wide Web at `http://satftp.soest.hawaii.edu/atlas/-#climate`

there, it decreases down to 5°C at a depth H_2 of about 700 m. Temperature further decreases down to an almost constant 1.5°C at $H_3 = 1000$m and deeper. (See Figure 9.)

An good model would be some sort of stretched sigmoid function, fit to points such as $(0, T_{surface})$, $(H_1, Tsurface)$, $(H_2, 5)$, $(H_3, 1.5)$. We attempted to use Maple V to fit that data to various functions, but it would only allow us to use polynomials, and not sigmoids. We also attempted to use a piece-wise defined function with each section of the curve approximated by a line. However, Maple was unable to solve equations with if-then statements in them. In the end, we had to settle for a polynomial. We had to add weights to the data, including several very deep points with temperature 1.5, to mold the graph into the right shape. It never did fit the data near the surface very well, and we were unable to determine how accurate the function was.

A.2   Obtaining the Submarine's Position from the Generalized Attenuation Coefficient

The next problem is how to determine the attenuation factor as a function of the total distance of a round trip. Consider one of the rays of sound as it travels from the surface down to the submarine, and then bounces to one of the two boats above. It is best to consider two parts of the path: from the surface to the submarine, and from the submarine to the boat.

On the way down to the submarine, the depth equals the distance travelled. (See Figure 10.) The attenuation of the amplitude A follows the differential equation

$$\frac{dA}{dd} = -\alpha(d)A(d)$$

where d is the depth of the sound. This is a separable differential equation whose solution may be written as

$$\frac{A_{bottom}}{A_{original}} = e^{\int_0^a -\alpha(d)\,dd} \tag{9}$$

where A_{bottom} is the amplitude at the submarine and $A_{original}$ is the sound's original amplitude.

For the second part of the sound's journey, it travels at an angle toward the boat. (See Figure 11.) Cosider a point moving along the line from the submarine to the boat. Let d be its depth, u be the distance from the
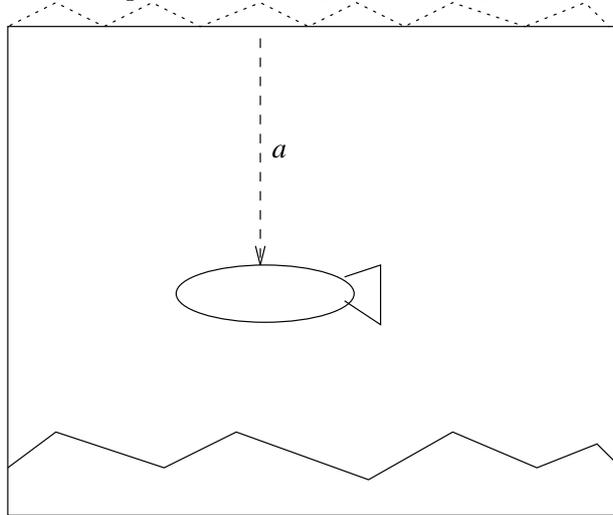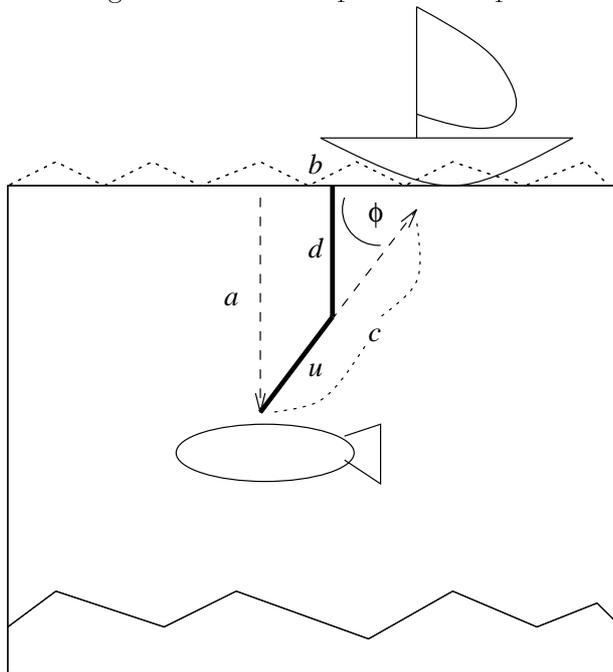
Figure 10: First part of the path

Figure 11: Second part of the path

submarine to the point, c be the total distance from the submarine to the boat, b be the distance along the top of the sea from the boat to a point directly above the submarine, and finally let ϕ be the angle between b and c.

The attenuation of sound still follows the usual rule, only this time, depth is a function of u:

$$\sin\phi = \frac{d}{c-u} \qquad \Longrightarrow \qquad d = (\sin\phi)(c-u) \tag{10}$$

so the differential equation becomes

$$\frac{dA}{dd} = -\alpha(d)A(d) = -\alpha((\sin\phi)(c-u))A(d)$$

This may be solved, again by separation of variables, giving

$$\frac{A_{final}}{A_{bottom}} = e^{\int_0^c -\alpha((\sin\phi)(c-u))\,du} \tag{11}$$

where A_{bottom} is the amplitude at the submarine and A_{final} is the sound's final amplitude when it reaches the boat.

The result of all this calculation is a system of three equations in a, c, and ϕ, which may (theoretically) be solved by some numerical method:

$$\frac{A_{final}}{A_{original}} = \left(e^{\int_0^c -\alpha((\sin\phi)(c-u))\,du} \right) \left(e^{\int_0^a -\alpha(d)\,dd} \right) \tag{12}$$

$$c^2 = a^2 + b^2 \tag{13}$$

$$\sin\phi = \frac{a}{c} \tag{14}$$

(Recall that b may be caluclated using information about the distance between the two boats and the angles at which the sound is detected.)

We made an attempt to enter these equations into Maple V and asked it to solve the system numerically. Sixty-nine megabytes later, we stopped it and gave up on finding a numerical example.

B   How Transducers Determine Direction

A transducer is a device which converts one type of energy to another. The transducers which are employed in this model convert pressure to voltage

and vice-versa, and the particular type of transducer which we have specified is called a piezoelectric crystal. Acoustic transducers are the most commonly used devices for underwater detection of sound in a scientific context, since they readily allow the sounds to be input into a circuit and filtered or separated by frequency. Once the sound is transformed into an alternating current in a circuit, it is a well-understood procedure to run Fourier analysis on the circuit to obtain frequency and amplitude information.

One cubical transducer by itself reacts to changes in pressure arriving generally along one or more of its axes. However, more than one transducer evenly spaced along a baseline can be made to react to pressure differences arriving only within a very specific angular range. A typical polar plot of sensitivity by angle is shown in Figure 12. This property of transducer arrays is called *directivity*. Suppose the sound is arriving in a line at a certain angle θ from the perpendicular to the array baseline. Since the incoming wavefront is nearly flat, it arrives at one end of the transducer array before it arrives at the other. Using the slight variations in arrival time of the wavefront, the circuitry behind the array can interpret the information it receives to figure out the direction of the sound. Then, by introducing signal-delay elements between each of the individual array elements and the circuit (See Figure 13), the array can be made to "sweep" through a range of angles with respect to the baseline's perpendicular. In this way, the array determines the direction of the incoming signal as well as the relative strengths of the incoming frequencies.
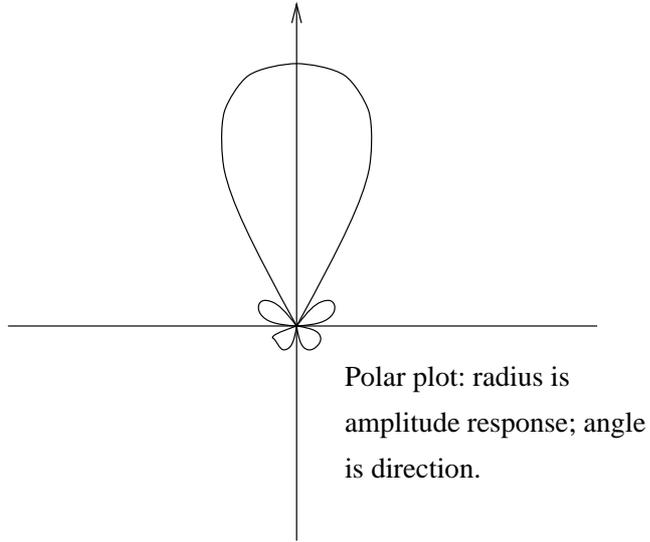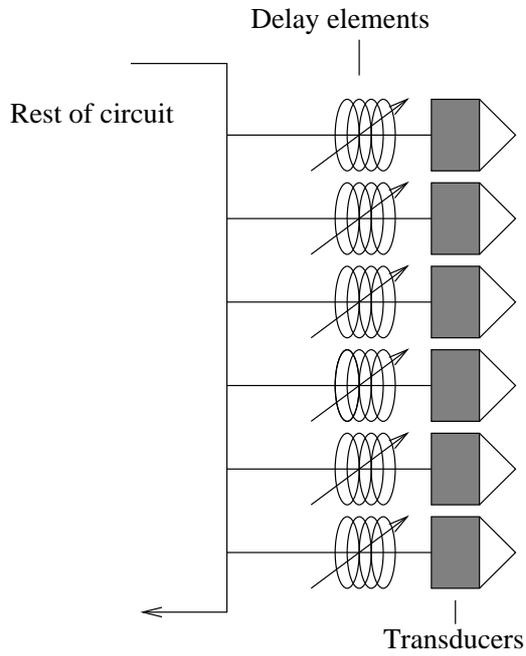
Figure 12: Polar Plot of Sensitivity by Direction

Polar plot: radius is amplitude response; angle is direction.

Figure 13: Transducer Array with Time Delays

Delay elements

Rest of circuit

Transducers

C   Calculating Errors

These calculations explain the results discussed in Section 2.8.

Suppose that θ_1 and θ_2 are measured to within $\pm\epsilon$. (Here, we will measure the angles in radians to make some calculations easier.) This error determines the accuracy of the calculations of b_1 and b_2. Recall the formulas for b_1:

$$\theta_3 = \pi - \theta_1 - \theta_2$$

$$b_1 = e\frac{\sin\theta_2}{\sin\theta_3}$$

If we now replace the angles by ranges:

$$\theta_3 = \pi - \theta_1 - \theta_2 \pm 2\epsilon$$

$$b_1 = e\frac{\sin(\theta_2 \pm \epsilon)}{\sin(\theta_3 \pm 2\epsilon)}$$

Expanding the sines of the sums and differences gives:

$$b_1 = e\frac{\sin\theta_2 \cos\epsilon \pm \cos\theta_2 \sin\epsilon}{\sin\theta_3 \cos 2\epsilon \pm \cos\theta_3 \sin 2\epsilon}$$

Since ϵ is small, we can approximate $\sin\epsilon \approx \epsilon$, $\sin 2\epsilon \approx 2\epsilon$ and $\cos\epsilon \approx \cos 2\epsilon \approx 1$. This simplification gives:

$$b_1 \approx e\frac{\sin\theta_2 \pm \epsilon\cos\theta_2}{\sin\theta_3 \pm 2\epsilon\cos\theta_3}$$

We can find minimum and maximum bounds for b_1 by first making the numerator minimal and the denominator maximal, and then maximizing the numerator and minimizing the denominator. It must also be noted that sin and cos range from $-1\ldots 1$, and that θ_1 and θ_2 range from $0\ldots\pi$ because they are interior angles of a triangle, and hence their sines are always non-negative. This reasoning yields the equations:

$$b_{1min} = e\frac{\sin\theta_2 - \epsilon}{\sin\theta_3 + 2\epsilon}$$

$$b_{1max} = e\frac{\sin\theta_2 + \epsilon}{\sin\theta_3 - 2\epsilon}$$

Finding upper and lower bounds for a is similar. Suppose that the transducers measure distances to within a certain percent δ. The distance d_1 will range from $d_1(1-\delta)\ldots d_1(1+\delta)$. Recall that:

$$a = \frac{d_1^2 - b_1^2}{2d_1}$$

To minimize a, we must minimize d_1 in the numerator, maximize b_1 in the numerator, and maximize d_1 in the denominator:

$$a_{min} = \frac{d_1^2(1-\delta)^2 - b_{1max}^2}{2d_1(1+\delta)}$$

To maximize a, we must maximize d_1, minimize b_1, and minimize d_1 in the denominator. So:

$$a_{max} = \frac{d_1^2(1+\delta)^2 - b_{1min}^2}{2d_1(1-\delta)}$$

This error analysis is rather general, and applies to any form of our model in which the angles from the boats to the submarines are accurate to within $\pm\epsilon$ and the distances derived from the sound attenuation are accurate to within some percent.

D   Measuring Sound in Decibels

Most microphones, or in this case, transducer arrays, measure sound intensity in terms of decibels (dB), which require some explanation.

The *intensity* (in units of W/m^2) of a plane wave travelling in a medium of density ρ at speed c, frequency f, and amplitude A is given by

$$I = \frac{1}{2}\rho c f^2 A^2 \tag{15}$$

In sea water, $\rho c = 1.5 \times 10^6 \frac{\text{kg}}{\text{m}^2\text{s}}$.

Decibels may be used to measure the ratio of two intensities, as in:

$$\text{dB} = 10\log_{10}\frac{I_1}{I_2}$$

To measure 'absolute' intensities in dB, the intensity is measured relative to the intensity I_0 of a sound with a root-mean-square pressure of 1μPa. A

sound's absolute intensity in dB is given by:

$$\text{dB re } 1\mu\text{Pa} = 10 \log_{10} \frac{I}{I_0}$$

The value of $I_0 = 0.67 \times 10^{-18} \frac{\text{W}}{\text{m}^2} = 0$ dB re 1μPa

If the dB measurment m of a sound is known, its intensity may be found with:

$$I = I_0 10^{\frac{m}{10}}$$

Transducers are likely to measure sound in either dB or units of intensity. However, it is important to know the amplitude of the sound to determine the location of the submarine, as is explained in Section 2.3. The amplitude may be determined from the intensity by solving equation 15 for A:

$$A = \sqrt{\frac{I}{\frac{1}{2}\rho c f^2}} \tag{16}$$

Where $\rho c = 1.5 \times 10^6 \frac{\text{kg}}{\text{m}^2\text{s}}$ in sea water.

E   Refraction and Detecting the Sea Floor

If the sea is too shallow and has too reflective a floor, our assumption that most of the sound field in the ocean originates at the surface and only decreases with depth may not be true. Taking the Atlantic Ocean as our example, and staying far from shore as per our assumptions, the depth of the ocean varies from 1500 m (at the Mid-Atlantic ridge) to 6000 m (at the Cape Verde Basin). A typical depth is about 4000 m. When the floor is basalt or another highly reflective substance, we can expect the sound waves hitting it to be reflected completely, while for clay or silt bottoms, most of the sound will be absorbed. A typical reflection ratio is 90%. For a 4000 m deep sea and a reflection ratio of 90% and no refraction, we can expect to receive about 89% of the surface noise back at the surface. That much reflection would make submarine detection very difficult.

Common sense says that the bottom does not reflect 89% of the sound from the surface. If it did, sound would reflect continuosly in the ocean, and the accumulated motion would make the sea too violent to support life.

We must therefore conclude that refraction at some point prevents surface noise from constantly bouncing off the ocean floor, and that the sound

eventually dissipates. This is why we assumed that refraction becomes a very serious factor beneath submarine depths. We may conclude that there will not be large amounts of sound reflecting off the sea floor and that it is unlikely that our model will mistake the ocean floor for a submarine.

F   Submarine or Whale?

Submarines are not the only large objects commonly found in deep seas. We can also expect to find whales, sunken ships, and so forth. Sunken ships need not be a major concern, since they will likely be found on the sea floor, and we have already accounted for ocean bottom echoes. How, then, shall we differentiate between whales and submarines?

The largest whales are about a fifth as long as the largest submarines, at 30 meters. They can dive as deep as 1000 meters on a long-term basis, which is twice as deep as submarines. Whales, however, reflect much less sound than steel vessels. In the final analysis, we must admit the possibility of detecting whales, but the reflected signals from the whales should be considerably diminished because of their low reflectivity and smaller size.

If a weak signal is detected and we wish to decide if it comes from a whale or a submarine, we can try monitoring the high frequency range. Submarines, according to the problem assumptions, are soundless. Whales, however, make high-pitched squeals and whistles. A boost in the high frequency range may indicate the presence of a whale.

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