Dynamics from ranking problems

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Abstract

This project is an exploration into (1) solving ranking problems with dynamical systems; (2) using ranking problems to formulate interesting dynamical systems; (3) what can dynamics say about rankability of a data set.

Ranking problem

Given a weighted digraph $\mathscr{G}$ with

- vertices $\mathscr{V} = \{1, \dots n\}$
- edges $\mathscr{E}$ of the form $j \to k$
- edge weights $w_{jk} \geq 0$

put the vertices into a linear order that is as consistent as possible with the data.

Formally: Given a weighted adjacency matrix W, choose a permutation σ such that the upper right triangle of $\sigma W \sigma^\mathsf{T}$ has maximum sum.

Example: Vertices are sports teams. Edge $j \to k$ means team j lost a game to team k. Weight w_{jk} is the difference in scores.

Basic dynamics

Assume each item j has a strength $q_j \in \mathbb{R}$. Goal is to eventually interpret strengths so that the probability that team j loses to k is an increasing function of $q_k - q_j$. Suggests trying to decrease this energy:

$$U(q) = \sum_{j,k} w_{jk} e^{q_j - q_k}$$

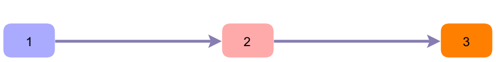
Seeking interesting dynamics, let us try Hamiltonian dynamics but with energy leak. Think of q_j as position of particle j, and let $p_j = \dot{q}_j$ be the corresponding momentum. Start with this Hamiltonian:

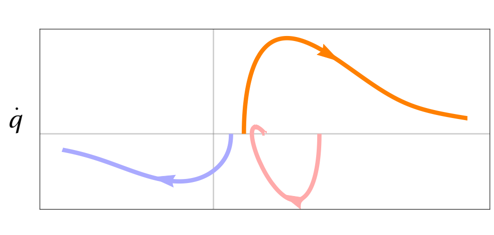
$$H_R = \sum_j \frac{p_j^2}{2} + U(q)$$

Equations of motion:

$$\ddot{q}_j + \overbrace{\beta \dot{q}_j}^{\text{friction}} + \frac{\partial U}{\partial q_j} = 0$$

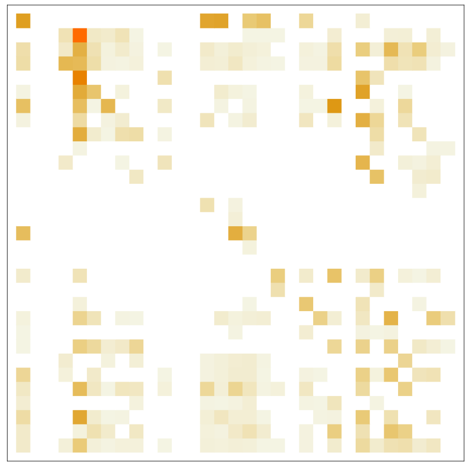
As time advances, particles scatter. Sort items by long-term order of q_j.





Example: LOLIB

From LOLIB, data set `N-t65l11xx`.
Original weight matrix Z:



Rescale for dynamics:

$$w_{jk} = \sqrt{Z_{jk}/z_{90}}$$

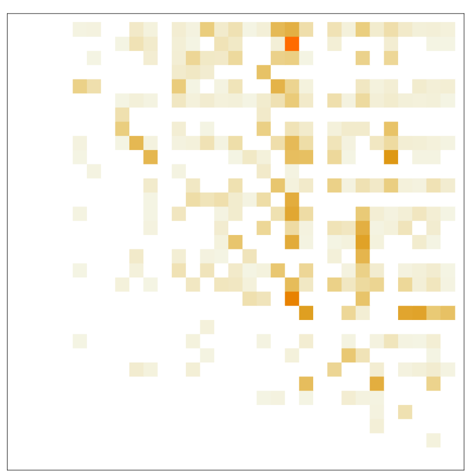
$$z_{90} = \text{90th percentile of entries of } Z$$

After the particles scatter, define permutation σ:

$$q_{\sigma(1)} < q_{\sigma(2)} < \cdots < q_{\sigma(n)}$$

re-order the weight matrix

$$\tilde{Z}_{jk} = Z_{\sigma(j)\sigma(k)}$$



Define the *tournament score*

$$\text{TS}(\tilde{Z}) = \sum_{k>j} \tilde{Z}_{jk}$$

For this data, $\text{TS}(\tilde{Z}) = 16357$, compared to the best known $\text{TS}_{\text{best}} = 16719$.

Results

On examples from LOLIB [Martí et al 2012], this produces reasonable results in a few seconds, about 90% of the best known tournament score.

LOLIB available at https://web.archive.org/web/20180702045352/http://www.optsicom.es/lolib/

Toda flow

From integrable systems theory, H_R looks very much like the Hamiltonian for the Toda chain. Traditional form:

$$H_T = \sum_j \frac{p_j^2}{2} + \sum_j e^{q_j - q_{j+1}}$$

Flaschka's change of variables:

$$b_j = -\frac{1}{2} p_j$$

$$a_j = \frac{1}{2} e^{(q_j - q_{j+1})/2}$$

Results in tri-diagonal matrices L and M that form a Lax pair:

$$\dot{L} = [M, L] \text{ and } H_T \propto \frac{1}{2} \operatorname{tr} L^2$$

As $t \to \infty$, p_j converges to a constant, particles scatter along asymptotically constant velocity trajectories in index order. My modification: Go beyond neighbor coupling, think of $a_{jk} \approx \frac{1}{2} e^{(q_j - q_k)/2}$.

$$A_{jk} = \begin{cases} \sqrt{w_{jk}} a_{jk} + i \sqrt{w_{kj}} a_{jk}^{-1} & \text{if } j < k \\ 0 & \text{otherwise} \end{cases}$$

$$B_{jk} = \begin{cases} b_j & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} L &= A + B + A^* \\ M &= A - A^* \\ \dot{L} &= [M, L] \end{aligned}$$

This reduces to the standard Toda chain when $\mathscr{G}$ is a chain with $w_{j,j+1} = 1$. When $\mathscr{G}$ is acyclic and the vertices are topologically sorted, $H_R \propto \frac{1}{2} \operatorname{tr} L^2$.

It was hoped that particles would scatter in some rank-like order. Flow is isospectral: Eigenvalues of L are constant. Toda flow accomplishes QR factorization [Chu, 1984]. In general, L converges to a diagonal matrix as $t \to \infty$, specifically, $b_j \to$ the j-th greatest eigenvalue of L. No ranking information is produced. Oh well.

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