Further Existence Results on Beautifully Ordered Balanced Incomplete Block Designs

Garrett Mitchener, Dinesh G. Sarvate, and Andrew Shealy

Abstract. Beautifully Ordered Balanced Incomplete Block Designs, BOBIBD$(v, k, \lambda, k_1, \lambda_1)$, were introduced by Chan and Sarvate along with some existence results for block size 3 and 4. We have shown that necessary conditions are sufficient for the existence of BOBIBDs with $k = 5$ for $k_1 = 2$ and 3 along with partial results for $k_1 = 4$. We also claim the nonexistence of cyclic solutions for certain BOBIBDs. The existence of the previously unknown BOBIBD$(v, 4, 2, 3, 1)$, $v \equiv 1 \pmod 6$, is demonstrated for all $v \geq 19$.

1. Introduction

Chan and Sarvate studied what is called *Beautifully Ordered Balanced Incomplete Block Designs* [**6**].

Definition 1. *If each of the blocks of a* BIBD(v, k, λ) *is ordered such that for any* $2 \leq k_1 \leq k$ *indices* $i_1, i_2, \cdots, i_{k_1}$ *the sub-blocks* $\{a_{i_1}, a_{i_2}, \cdots, a_{i_{k1}}\}$ *of all ordered blocks* $\{a_1, a_2, \cdots, a_k\}$ *of the* BIBD(v, k, λ) *form a* BIBD(v, k_1, λ_1) *then we say that the collection of ordered blocks gives a Beautifully Ordered Balanced Incomplete Block Design,* BOBIBD$(v, k, \lambda, k_1, \lambda_1)$ *where* $2 \leq k_1 \leq k\text{-}1$.

1.1. Necessary Conditions for BOBIBDs.

From the definition, if a BOBIBD$(v,k,\lambda,k_1,\lambda_1)$ exists, then

(1) BIBD(v,k,λ) exists, and
(2) BIBD(v,k_1,λ_1) exists.

Hence:

Key words and phrases. BOBIBD.

THEOREM 1. *Every necessary condition for the existence of* BIBD(v,k,λ) *is a necessary condition for the existence of* BOBIBD(v,k,λ,k_1,λ_1) *and every necessary condition for the existence of* BIBD(v,k_1,λ_1) *is a necessary condition for the existence of* BOBIBD(v,k,λ,k_1,λ_1).

The well known [**7**] necessary conditions for BIBD(v,3,λ), BIBD(v,4,λ) and BIBD(v, 5, λ), for $v \geq k$, are necessary for the purpose of this note. For ease of reference they are given below:

Block size 3:

λ	spectrum of BIBD$(v, 3, \lambda)$
$\lambda \equiv 0 (\text{mod } 6)$	all $v \neq 2$
$\lambda \equiv 1,5 (\text{mod } 6)$	all $v \equiv 1,3 (\text{mod } 6)$
$\lambda \equiv 2,4 (\text{mod } 6)$	all $v \equiv 0,1 (\text{mod } 3)$
$\lambda \equiv 3 (\text{mod } 6)$	all odd v

Block size 4:

λ	spectrum of BIBD$(v, 4, \lambda)$
$\lambda \equiv 0 (\text{mod } 6)$	all v
$\lambda \equiv 1,5 (\text{mod } 6)$	all $v \equiv 1,4 (\text{mod } 12)$
$\lambda \equiv 2,4 (\text{mod } 6)$	all $v \equiv 1 (\text{mod } 3)$
$\lambda \equiv 3 (\text{mod } 6)$	all $v \equiv 0,1 (\text{mod } 4)$

Block size 5:

λ	spectrum of BIBD$(v, 5, \lambda)$
$\lambda \equiv 0 (\text{mod } 20)$	all v
$\lambda \equiv 1,3,7,9,11,13,17,19 (\text{mod } 20)$	all $v \equiv 1,5 (\text{mod } 20)$
$\lambda \equiv 2,6,14,18 (\text{mod } 20)$	all $v \equiv 1,5 (\text{mod } 10)$
$\lambda \equiv 4,8,12,16 (\text{mod } 20)$	all $v \equiv 0,1 (\text{mod } 5)$
$\lambda \equiv 5,15 (\text{mod } 20)$	all $v \equiv 1 (\text{mod } 4)$
$\lambda \equiv 10 (\text{mod } 20)$	all $v \equiv 1 (\text{mod } 2)$

.

Simple counting arguments give:

THEOREM 2. *In a* BOBIBD(v,k,λ,k_1,λ_1), $\lambda = \frac{\binom{k}{k_1}\lambda_1}{\binom{k-2}{k_1-2}}$.

Given a BOBIBD(v, k, λ,2,λ_1), there are $\binom{k}{2}$ ways we can pick up two locations in a block of BIBD(v, k, λ). Hence, λ must be $\binom{k}{2}\lambda_1$, a multiple of $\binom{k}{2}$. This fact is included in the following corollary.

COROLLARY 1. *For any BOBIBD,*

(1) *if* $k_1 = 2$, *then* $\lambda = \binom{k}{2}\lambda_1$ *and the number of blocks must be a multiple of* $\binom{v}{2}$.
(2) *if* $k_1 = 3$, *then* $\lambda = \frac{(\binom{k}{3}\lambda_1)}{(k-2)} = \frac{\binom{k}{2}\lambda_1}{3}$.

(3) *if* $k_1 = 4$, *then* $\lambda = \frac{(\binom{k}{4}\lambda_1)}{\binom{k-2}{2}} = \frac{\binom{k}{2}\lambda_1}{6}$.

This follows immediately from Theorem 2.

THEOREM 3. *If a* BOBIBD(v,k,λ,2,λ_1) *exists, then a* BOBIBD(v,k,λ, k_1,$\binom{k_1}{2}\lambda_1$) *exists for* $2 \leq k_1 \leq k$.

Note that the converse is not true as shown below.

EXAMPLE 1. *A* BOBIBD(4,4,4,3,2) *with blocks* {1,2,3,4}, {4,1,2,3}, {3,4,1,2}, {2,3,4,1} *is not a* BOBIBD(4,4,4,2,1). *In fact a* BOBIBD(4,4,4,2,1) *does not exist.*

In view of the above theorem, all results and examples obtained for $k =$ 4 and $k_1 = 2$ extend for $k = 4$ and $k_1 = 3$ as well as a BOBIBD$(v, 4, \lambda, 2, \lambda_1)$ is also a BOBIBD$(v, 4, \lambda, 3, 3\lambda_1)$.

THEOREM 4. *If a* BOBIBD(v,k,λ,2,λ_1) *exists, then a* BOBIBD(v,k_1, $\binom{k_1}{2}\lambda_1$,2,λ_1) *exists, where* $2 \leq k_1 \leq k$.

The following theorem is an important result.

THEOREM 5. *If a* BOBIBD$(v, k, \lambda, k_1, \lambda_1)$ *exists then* k *divides* r *and in the* (*ordered*) *blocks of the* BOBIBD *each element occurs exactly* $\frac{r}{k}$ *times at each location of the blocks.*

EXAMPLE 2. *For the* BIBD(4,4,6), $r = 6$ and $\frac{r}{k} = \frac{6}{4}$, which is not an integer. Therefore BOBIBD(4,4,6,2,1) does not exist .

Proofs for the above results are given in [**6**] where the authors showed that necessary conditions are sufficient for the existence of BOBIBDs with block size $k = 3$ and $k = 4$ for $k_1 = 2$. Existence of BOBIBDs with block size $k = 4$ and $k_1 = 3$ is demonstrated for all but one infinite family and the non-existence of BOBIBD$(7, 4, 2, 3, 1)$, the first member of the unknown series, is shown in [**6**]. Professor Stinson informed us of his paper [**10**] where the existence of the unknown series BOBIBD$(v, 4, 2, 3, 1)$ is settled. We prove the existence of BOBIBD$(v, 5, \lambda, k_1, \lambda_1)$ for $k_1 = 2, 3$, but $k_1 = 4$ is still open.

1.2. Pairwise Balanced Designs. Let K be a subset of positive integers and let λ be a positive integer. A *pairwise balanced design*, PBD$(v, K; \lambda)$, of order v with block sizes from K is a pair $(\mathcal{V}, \mathcal{B})$, where $\mathcal{V}$ is a finite set with cardinality v and $\mathcal{B}$ is a family of subsets (called blocks) of $\mathcal{V}$ which satisfy the following properties:

(1) If $B \in \mathcal{B}$, then the cardinality of B is an element of K.
(2) Every pair of distinct elements of $\mathcal{V}$ occurs in λ blocks of B.

When $\lambda = 1$ the notation PBD(v, K) is used.

A set $\mathcal{S}$ of positive integers is *PBD-closed* if the existence of a PBD$(v, \mathcal{S})$ implies that v belongs to $\mathcal{S}$. Let K be a set of positive integers and let B(K) := $\{v \mid \exists$ PBD$(v, K)\}$. Then B(K) is the *PBD-closure* of K. If $K = \{k\}$, the notation B(k) is used and a PBD(v, K) is a BIBD(v, k, λ). There are many existence results on the PBD-closure found in [**3**].

LEMMA 1. *If* PBD (v, K) *exists and for all* $k_t \in K$ *a* BIBD(k_t, k, λ) *exists for some integer* k*, then we can construct a* BIBD$(v, k, 1)$.

THEOREM 6. *If* PBD(v, K) *exists and for all* $k_t \in K$ *a* BOBIBD$(k_t, k, \lambda, k_1, \lambda_1)$ *exists for some integer* k*, then a* BOBIBD$(v, k, \lambda, k_1, \lambda_1)$ *exists.*

PROOF. Let $k_t \in K$. Replacing each block $\{x_1, x_2, \ldots, x_{k_t}\}$ of PBD(v, K) by BOBIBD$(k_t, k, \lambda, k_1, \lambda_1)$ on the set of elements $\{x_1, x_2, \ldots, x_{k_t}\}$ we obtain a BOBIBD$(v, k, \lambda, k_1, \lambda_1)$. □

There are many existence results on the closure of subsets on pairwise balanced designs found in [**3**]. We list below the ones needed for our purpose.

Subset	Closure	Genuine Exceptions
5 9	1 mod 4	13 17 29 33 113
5 9 13	1 mod 4	17 29 33
5 6 7 8 9	$\mathbb{N}$	(10-20) (22-24) (27-29) (32-34)

Table 1

2. Block sizes $k = 4$, $k_1 = 2$

From Theorem 2, if BOBIBD$(v, 4, \lambda, 2, \lambda_1)$ exists then $\lambda = 6\lambda_1$. The necessary conditions for the existence of BOBIBD$(v, 4, \lambda, 2, \lambda_1)$ are:

λ	Spectrum of BOBIBD$(v, 4, \lambda, 2, \lambda_1)$'s
$\lambda \equiv 6 (\text{mod } 12)$	all odd $v \geq 5$
$\lambda \equiv 0 (\text{mod } 12)$	no condition on v

For $\lambda = 6$, Theorem 12 in [**6**] shows the necessary conditions for the existence of a BOBIBD$(v, 4, 6t, 2, t)$ are sufficient, except possibly for $15, 27, 33, 39, 51, 75, 87, 95, 99, 111$, and 115. We observe that perpendicular arrays give BOBIBDs for these exceptions.

DEFINITION 2. *A perpendicular array* PA$_\lambda(t, k, v)$ *is a* $k \times \lambda\binom{v}{t}$ *array where each entry is from* $\{1, 2, \cdots, v\}$ *such that*

1. *each column has* k *distinct entries, and*
2. *each set of* t *rows contains each set of* t *distinct entries as a column precisely* λ *times.*

When $t = 2$ the perpendicular array gives a BOBIBD with $k_1 = 2$ when we consider the columns of the array as the blocks of the BIBD(v, k, λ). Furthermore,

THEOREM 7. *Perpendicular array* $\mathrm{PA}_{\lambda_1}(2, k, v)$ *exists if and only if a* $\mathrm{BOBIBD}(\mathrm{v}, \mathrm{k}, \lambda_1\binom{\mathrm{k}}{2}, 2, \lambda_1)$ *exists.*

PROOF. Given the perpendicular array $\mathrm{PA}_{\lambda_1}(2, k, v)$, we treat the columns of the array as blocks. By definition of a perpendicular array, any two rows give a BIBD$(v, 2, \lambda_1)$ exist. Since we are treating the columns as blocks and and there are k rows in the $\mathrm{PA}_{\lambda_1}(2, k, v)$, there are $\binom{k}{2}$ BIBD$(v, 2, \lambda_1)$'s. Hence we have the existence of the BOBIBD$(v, k, \lambda_1\binom{k}{2}, 2, \lambda_1)$.

Given a $\mathrm{BOBIBD}(\mathrm{v}, \mathrm{k}, \lambda_1\binom{\mathrm{k}}{2}, 2, \lambda_1)$. Construct an array where the columns represent the blocks of the BIBD$(v, k, \lambda_1\binom{k}{2})$. From definition of BOBIBD$(v, k, \lambda_1\binom{k}{2}, 2, \lambda_1)$, picking any two rows of the array gives a BIBD$(v, 2, \lambda_1)$. Thus, each 2 rows contains each set of 2 distinct entries as a column (pairs) precisely λ_1 times and b $= \lambda_1\binom{v}{2}$ as required.

□

Note that when $k_1 \geq 3$, perpendicular arrays and BOBIBDs are two different mathematical structures, the perpendicular arrays are not concerned with pairs, whereas BOBIBDs are. There are many existence results on perpendicular arrays as given in [**5**]. It is given in [**5**] that:

- BOBIBD$(v, 4, 6, 2, 1)$ exists for all odd $v \geq 5$ [**9**], [**8**]

Hence, for all exceptions shown in Theorem 12 in [**6**], the BOBIBD$(v, 4, 6, 2, 1)$ exist.

THEOREM 8. *Necessary conditions are sufficient for the existence of* BOBIBD$(v, 4, 6, 2, 1)$ *for all odd* $v \geq 5$.

3. Block sizes $k = 4, k_1 = 3$

The existence of the BOBIBD$(v, 4, 2, 3, 1)$ for $v \equiv 1 (\mathrm{mod}\ 6)$ was dealt with in [**6**]. The authors showed the nonexistence of the first member of the family, BOBIBD$(7, 4, 2, 3, 1)$, by brute force. A computer program confirmed that BOBIBD$(7, 4, 2, 3, 1)$ cannot be developed using difference sets. The proof of the following theorem can be found in [**6**].

THEOREM 9. *The blocks of a* BIBD$(v, 4, 2)$ *can not be ordered to contruct a* BOBIBD$(v, 4, 2, 3, 1)$ *if there exist two identical blocks or two blocks with* 3 *common points.*

A computer program showed that BOBIBD$(13, 4, 2, 3, 1)$ cannot be constructed using difference sets, and it ensured the existence of BOBIBD$(19, 4, 2, 3, 1)$ through difference sets.

Theorem 10. BOBIBD$(19, 4, 2, 3, 1)$ *exist by construction of the following ordered difference sets:*

$$\{0, 1, 4, 5\}, \{9, 2, 0, 12\}, \{2, 13, 8, 0\} \textit{ modulo } 19.$$

D.R. Stinson informed us of one of his papers [**10**]. In [**10**] the following definition is presented along with existence results.

Definition 3. *A perpendicular array of triple systems of order v (or a* PATS(v)*), is a $\frac{1}{6}v(v-1)$ by 4 array T (of points chosed from a set X of size v) such that:*

(1) *the rows of T form a* BIBD$(v, 4, 2)$*;*
(2) *For any subarray T' consisting of 3 columns of T, the rows of T' form an* STS(v) *(a steiner triple system of order v).*

It is clear to see that this definition and the definition of a BOBIBD$(v, 4, 2, 3, 1)$ are the same. The main result for Stinson's paper [**10**] was that PATS(v) exists for all $v \equiv 1 (\text{mod } 6)$, $v \geq 19$, with four possible exceptions $v = \{43, 55, 85, 133\}$. Therefore,

Theorem 11. *Necessary conditions are sufficient for the existence of* BOBIBD$(v, 4, 2, 3, 1)$ *for $v \geq 19$, with four possible exceptions $v = \{43, 55, 85, 133\}$.*

4. Block sizes $k = 5$, $k_1 = 2$

From Theorem 2, if a BOBIBD$(v, 5, \lambda, 2, \lambda_1)$ exist then $\lambda = 10\lambda_1$. The necessary conditions for the existence of BOBIBD$(v, 5, \lambda, 2, \lambda_1)$ are:

λ	spectrum of BOBIBD$(v, 5, \lambda, 2, \lambda_1)$'s
$\lambda \equiv 0(\text{mod } 20)$	no condition on v
$\lambda \equiv 10(\text{mod } 20)$	all odd $v \geq 5$

4.1. v **odd.** Let v be odd. Letting $\lambda = 1$, we have from [**5**] and Theorem 7 that BOBIBD$(v, 5, 10, 2, 1)$ exists for all odd $v \geq 5$ and $v \neq 39$ by the use of perpendicular arrays.

Theorem 12. *Necessary conditions are sufficeint for the existence of a* BOBIBD$(v, 5, 10, 2, 1)$ *for all odd $v \geq 5$ and $v \neq 39$.*

4.2. v **even.** Let $v = 2m$, $m \in \mathbb{Z}^+$. From the spectrum of λ-fold quintuple systems $\lambda \equiv 0(\text{mod } 20)$. Thus, for some $s \in \mathbb{Z}$, $\lambda = 20s$, we are concerned with the construction of BOBIBD$(2m, 5, 20s, 2, 2s)$.

We will need the following well known results about Latin squares. For basic definition and notation, please see [**11**]. A Latin square L of order v on symbols $Q = \{1, 2, \cdots, v\}$ is an $v \times v$ array in which each element of Q occurs in each row and each column exactly once. One may consider a

Latin square as a quasigroup $(Q, \circ)$ by labeling the rows and the columns of L by the elements of Q where $i \circ j$ is the $(i,j)^{th}$ element, the element in the i^{th} row and j^{th} column of L. When $(i,i)^{th}$ element of L is i for all $i = 1, 2, \cdots, v$, L is called an idempotent Latin square. Let $N(v)$ denote the number of Latin squares in the largest possible set of mutually orthogonal Latin squares of order v.

THEOREM 13. [**11**] *If $q = p^e$ is a prime power then $N(q) = q - 1$.*

LEMMA 2. [**11**] *There exists a set of $N(v)$ - 1 mutually orthogonal idempotent Latin squares of order v.*

THEOREM 14. [**11**] *There exist three mutually orthogonal Latin squares of every order except* 2, 3, 6, *and possibly* 10.

COROLLARY 2. [**11**] *There is a pair of orthogonal idempotent Latin squares of every side except* 2, 3, 6 *and possibly 10.*

THEOREM 15. *Let $L_1 = (Q, \circ_1)$, $L_2 = (Q, \circ_2)$, $\ldots$ $L_{(k-2)} = (Q, \circ_{(k-2)})$ be $k-2$ mutually orthogonal idempotent Latin squares of order v. Then the set of blocks $\mathcal{B} = \{\{a,b,a\circ_1 b,a\circ_2 b,\ldots,a\circ_{(k-2)} b\} : a \neq b \ a,b \in Q\}$ gives a* BOBIBD$(v, k, 2\binom{k}{2}, 2, 2)$.

PROOF. Note that the number of blocks of size k in $\mathcal{B}$ is $2\binom{v}{2} = v(v-1)$, same as the required number of blocks for a BIBD$(v, k, 2\binom{k}{2})$. For any $m \neq n$, we know that pair $\{m,n\}$ occurs at the first two locations of the blocks exactly twice by our construction of blocks: once as $\{m,n\}$ and once as $\{n,m\}$. Now consider the occurrences of the pair $\{m,n\}$ at the first and i^{th} location, where $i > 2$. For some $x, y \in Q$, $m\circ_{(i-2)} x = n$ and $n\circ_{(i-2)} y = m$. The pair $\{m,n\}$ appears at location $(1,i)$ in the blocks at least twice, once as $\{m, x, \ldots, n, \ldots\}$ and once as $\{m, x, \ldots, n, \ldots\}$. A similar argument shows that pair $\{m,n\}$ appear at least twice at entry $(2,i)$, $i > 2$. Next we consider the occurrences of the pair $\{m,n\}$ at the i^{th} and j^{th} location, where $i,j > 2$ and $i \neq j$. As the Latin squares L_i and L_j are orthogonal, pair $\{m,n\}$ and $\{n,m\}$ appear in location $\{i,j\}$ of the blocks exactly once. Hence, $\{m,n\}$ appears at location $\{i,j\}$ exactly twice. Now, since every pair occurs at any two distinct entries at least twice and since there are exactly $2\binom{v}{2}$ blocks, we have that each pair occurs exactly twice at any two distinct entries. This gaurentees us the existence of BOBIBD$(v, k, \lambda, 2, 2)$. Since each pair is occuring at any two distinct locations and since there are $\binom{k}{2}$ distinct locations, we have that $\lambda = 2\binom{k}{2}$. Consequently, a BOBIBD$(v, k, 2\binom{k}{2}, 2, 2)$ exists.

□

In [**2**] three mutually orthogonal idempotent Latin squares of orders 22 and 26 are shown. There exist three mutually orthogonal idempotent

Latin squares of order $v = 8, 12, 16, 18, 20, 24, 28, 32, 34$ as shown in [**1**]. Consequently,

COROLLARY 3. BOBIBD$(v, 5, 20, 2, 2)$ *exist for* $v = 8, 12, 16, 18, 20, 22, 24, 26, 28, 32$ *and* 34.

We will now see that the converse of Theorem 15 does not hold. Theorem 14 shows that there does not exist 3 idempotent MOLS for $v = 6$. Thus, one would suspect a BOBIBD$(6, 5, 20, 2, 2)$ does not exist. However, this is not so.

LEMMA 3. BOBIBD$(6, 5, 20, 2, 2)$ *exist by construction of difference family* $\{3, 1, 4, 2, 0\}, \{\infty, 3, 4, 2, 1\}, \{1, \infty, 3, 4, 2\}, \{2, 1, \infty, 3, 4\}, \{4, 2, 1, \infty, 3\}$ $\{3, 4, 2, 1, \infty\}$, *modulo* 5.

It is known for all v a PBD$(v, \{5, 6, 7, 8, 9\})$ exist, and as shown previously, BOBIBD$(v, 5, \lambda, 2, \lambda_1)$ exist for $v = 5, 6, 7, 8$ and 9. Thus, from Theorem 6 and Table 1, BOBIBD$(v, 5, \lambda, 2, \lambda_1)$ exist for all v except possibly $v = (10-20), (22-24), (27-29), (32-34)$. However, we are left to check the outcome for $v = 10, 14$ for a complete result. It is shown in [**4**] that PA$_2(2, 5, 10)$ and PA$_2(2, 5, 14)$ exist due to the existence of APA$_2(2, 5, 10)$ and APA$_2(2, 5, 14)$, respectively. Consequently,

THEOREM 16. *Necessary conditions are sufficient for the existence of a* BOBIBD$(v, 5, \lambda, 2, \lambda_1)$ *for all* $v \geq 5$ *and* $v \neq 39$.

5. Block sizes $k = 5$, $k_1 = 3$

Simple counting arguments give that $\lambda = \frac{10\lambda_1}{3}$ for a BOBIBD$(v, 5, \lambda, 3, \lambda_1)$ to exist. Note that λ_1 is a multiple of 3. The necessary conditions for the existence of BOBIBD$(v, 5, \lambda, 3, \lambda_1)$ are:

λ	spectrum of BOBIBD$(v, 5, \lambda, 3, \lambda_1)$'s
$\lambda \equiv 0 (\text{mod } 20)$	no condition on v
$\lambda \equiv 10 (\text{mod } 20)$	all odd $v \geq 5$

5.1. v **odd.** Let v be odd. We would like to know the outcome of the family BOBIBD$(v, 5, 10t, 3, 3t)$, where $t \in \mathbb{Z}$. From Theorem 3 since BOBIBD$(v, 5, 10t, 2, t)$ exist for all odd $v \geq 5$ with $v \neq 39$, then BOBIBD$(v, 5, 10t, 3, 3t)$ exist for all odd $v \geq 5$ with $v \neq 39$, as well.

COROLLARY 4. BOBIBD$(v, 5, \lambda, 3, \lambda_1)$ *exist for all odd* $v \geq 5$ *except possibly* $v = 39$.

For $v = 39$, the underlying BIBDs exist and the necessary conditions are satisfied; therefore one might expect a BOBIBD$(39, 5, 10, 3, 3)$ to exist. However, the existence of BOBIBD$(39, 5, 10, 3, 3)$ is still unknown.

5.2. v **even.** Let v be even. From the spectrum of λ-fold quintuple systems, we have that $\lambda \equiv 0 (\text{mod } 20)$. Thus, for some $s \in \mathbb{Z}$, $\lambda = 20s$ and also, $\lambda_1 = 6s$. From Theorem 3, since we have that for all even $v > 5$ the family BOBIBD$(v, 5, 20s, 2, 2s)$ exist, then the family BOBIBD$(v, 5, 20s, 3, 6s)$ exist for all even $v > 5$, as well. With this, we have exhausted all possible values for v except possibly $v = 39$.

Theorem 17. *Necessary conditions are sufficient for the existence of a* BOBIBD$(v, 5, \lambda, 3, \lambda_1)$ *for all* $v \geq 5$*, except possibly* $v = 39$.

6. Block sizes $k = 5, k_1 = 4$

We will determine the family BOBIBD$(v, 5, \lambda, 4, \lambda_1)$. From the necessary conditions λ is a multiple of 5. We can obtain some BOBIBDs for $k_1 = 4$ from BOBIBDs for $k_1 = 2$. The necessary conditions for the existence of BOBIBD$(v, 5, \lambda, 4, \lambda_1)$ are:

λ	spectrum of BOBIBD$(v, 5, \lambda, 4, \lambda_1)$'s
$\lambda \equiv 0 (\text{mod } 20)$	no condition on v
$\lambda \equiv 5 (\text{mod } 20)$	all $v \equiv 1 (\text{mod } 4)$
$\lambda \equiv 10 (\text{mod } 20)$	all odd $v \geq 5$
$\lambda \equiv 15 (\text{mod } 20)$	all $v \equiv 1 (\text{mod } 4)$

Given BOBIBD$(v, 5, 10t, 2, t)$ a BOBIBD$(v, 5, 10t, 4, \binom{4}{2}t)$ exists. Thus, the family BOBIBD$(v, 5, 10t, 4, 6t)$ for $\lambda \equiv$ 0,10 (mod 20) is known from Theorem 16.

Lemma 4. *A* BOBIBD$(v, 5, 10t, 4, 6t)$ *exist for all* $v \geq 5$*, except possibly* $v = 39$.

We are left to find the family of BOBIBD$(v, 5, \lambda, 4, \lambda_1)$ for $\lambda \equiv$ 5,15 (mod 20).

6.1. $\lambda \equiv$ **5,15 (mod 20).** For when $\lambda = 5t$, we have that $\lambda_1 = 3t$. We began with an example.

Example 3. *Observe* BOBIBD$(5, 5, 5, 4, 3)$

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

Consequently, we have the first member for the family BOBIBD$(v, 5, 5t, 4, 3t)$.

A BIBD$(v, 5, 1)$ exists if $v \equiv$ 1,5 (mod 20), which is the same for $v \equiv 1 (\text{mod } 4)$. For BIBD$(v, 5, 1)$ cycle each block as we did in the previous example and the outcome will give us a BOBIBD$(v, 5, 5, 4, 3)$, which in turn will give us a BOBIBD$(v, 5, 15, 4, 9)$ when we replicate each block three times.

From Table 1, if BOBIBD$(v, 5, 5, 4, 3)$ for $v = 5$ and 9 exists then a BOBIBD$(v, 5, 5, 4, 3)$ exist for all v except possibly $v = 13, 17, 29, 33, 113$. A computer program shows that a BOBIBD$(9, 5, 5, 4, 3)$ cannot be constructed via difference sets in $\mathbb{Z}_9$ or in $\mathbb{Z}_3 \times \mathbb{Z}_3$. Also, the following must hold true for a BOBIBD$(9, 5, 5, 4, 3)$:

THEOREM 18. *For* BOBIBD$(9, 5, 5, 4, 3)$ *there cannot exist four or more blocks with three common points.*

PROOF. Assume, to the contrary, there exist four blocks with three common points. Without loss of generality, suppose these blocks, listed columnwise, are as follows:

1	1	1	1
2	2	2	2
3	3	3	3

The replication number $r = 10$ for BIBD$(9, 5, 5)$. Assume,

1	1	1	1	1	1	1	1	1	1
2	2	2	2						
3	3	3	3						

Since $\lambda = 5$, each pair needs to occur 5 times. Thus, pair $\{1, 2\}, \{1, 3\}$ and $\{2, 3\}$ all need to occur 5 times. Thus, the following is assumed:

1	1	1	1	1	1	1	1	1	1	2
2	2	2	2	2	3					3
3	3	3	3							

Again, since $r = 10$ elements 2 and 3 need to occur 10 times. Thus,

1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	3	3	3	3
2	2	2	2	2	3					3								
3	3	3	3															

This shows us that $b = 19$. Which is a contradiction because for a BIBD$(9, 5, 5)$, $b = 18$. □

A computer program ensures us the existence of BOBIBD$(13, 5, 5, 4, 3)$ through difference sets.

Lemma 5. BOBIBD$(13, 5, 5, 4, 3)$ *exist by construction of difference family* $\{0, 1, 3, 5, 6\}$, $\{5, 3, 9, 2, 0\}$, $\{2, 6, 1, 9, 0\}$ *modulo* 13.

From Table 1, if BOBIBD$(9, 5, 5, 4, 3)$ exists then a BOBIBD$(v, 5, 5, 4, 3)$ exists for all v except possibly $v = 17, 29, 33$. Therefore, the existence of BOBIBD$(9, 5, 5, 4, 3)$ is important.

References

[1] R.J.R Abel, A.E. Brouwer, C.J. Colburn and J.H. Dinitz, *Mutually Orthogonal Latin Squares (MOLS)*, The Handbook of Combinatorial Designs, first edition, edited by C. J. Colbourn and J. H. Dinitz, CRC Press, Boca Raton, Fl., 1996, 114–119.

[2] R.J.R. Abel, Z. Xiafu and Z. Hangfu, *Three mutually orthogonal idempotent Latin squares of orders 22 and 26*, Journal of Statistical Planning and Inference, **51** Issue 2 (1996), 101–106.

[3] F.E. Bennett, H.-D.O.F. Gronau, A.C.H. Ling and R.C. Mullin, *PBD-Closure*, The Handbook of Combinatorial Designs, first edition, edited by C. J. Colbourn and J. H. Dinitz, CRC Press, Boca Raton, Fl., 1996, 203–213.

[4] Jürgen Bierbrauer, *Ordered Designs and Perpendicular Arrays*, The Handbook of Combinatorial Designs, first edition, edited by C. J. Colbourn and J. H. Dinitz, CRC Press, Boca Raton, Fl., 1996, 396–399.

[5] Jürgen Bierbrauer, *Ordered designs, perpendicular arrays and permutation sets*, The Handbook of Combinatorial Designs, second edition, edited by C. J. Colbourn and J. H. Dinitz, Chapman/CRC Press, Boca Raton, Fl., 2007, 543–547.

[6] Hau Chan and D.G. Sarvate, *Beautifully Ordered Balanced Incomplete Block Designs*, J. Combin. Math. Combin. Comput. **74** (2010), 65–81.

[7] Haim Hanani, *Balanced Incomplete Block Designs and Related Designs*, Discrete Mathematics **11** (1975), 255–369.

[8] C. C. Lindner, R. C. Mulllin, and G. H. J. Van Rees, *Separable orthogonal arrays*, Utilitas Math. **31** (1987), 25–32.

[9] R.C. Mullin, P. J. Schellenberg, G. H. J. Van Rees, and S. A. Vanstone, *On the construction of perpendicular arrays*, Utilitas Math. **18** (1980), 141–160.

[10] D.R. Stinson, *Concerning the Spectrum of Perpendicular Arrays of Triple Systems*. Discrete Mathematics **61** (1986), 305–310.

[11] W.D. Wallis, *Introduction to Combinatorial Designs*, Second edition, Chapman and Hall/CRC, 2007.

(A. One) College of Charleston, Dept. of Math., Charleston, SC, 29424
E-mail address: `mitchenerg@cofc.edu`

(A. Two) College of Charleston, Dept. of Math., Charleston, SC, 29424
E-mail address: `sarvated@cofc.edu`

(A. Three) College of Charleston, Dept. of Math., Charleston, SC, 29424
E-mail address: `akshealy@g.cofc.edu`