Ranking with Hamiltonian dynamics

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September 6, 2020

Ranking problems

Given

- Items $x_1, x_2, \ldots x_n$,
- Weighted comparison matrix W
  Entry w_{jk} indicates how strongly x_j should be placed before x_k

find a permutation τ that gives a linear ordering

$$x_{\tau(1)}, x_{\tau(2)}, \ldots$$

that is as consistent as possible with W

Solution methods

- Brute force
- Heuristics
- Integer program
- RankBoost
- Dynamical systems

Goals

- Develop theory and practical algorithms for ranking
- Develop notion of *rankability*
- Develop interesting dynamical systems

Ranking potential

$$R(q; W, \gamma_{\mathrm{r}}) = \sum_{j,k} w_{jk} e^{\gamma_{\mathrm{r}} \cdot (q_j - q_k)}$$

- q_j: position of particle j
- γ_{r}: scaling parameter

Seek system states such that $R(q)$ is low

Ranking potential

$$\overbrace{R(q; W, \gamma_{\mathrm{r}})}^{\text{empirical risk}} = \sum_{j,k} \underbrace{w_{jk} e^{\gamma_{\mathrm{r}} \cdot (q_j - q_k)}}_{\text{loss}}$$

- q_j: position of particle j
- γ_{r}: scaling parameter

Seek system states such that $R(q)$ is low

Hamiltonian dynamics

Confinement potential

$$C(q;\gamma_c) = \sum_j \cosh\left(\gamma_c q_j\right)$$

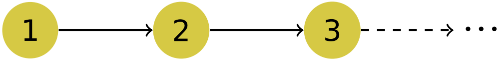
Hamiltonian

$$H = \frac{1}{2}\sum_j p_j^2 + \alpha_r R(q) + \alpha_c C(q)$$

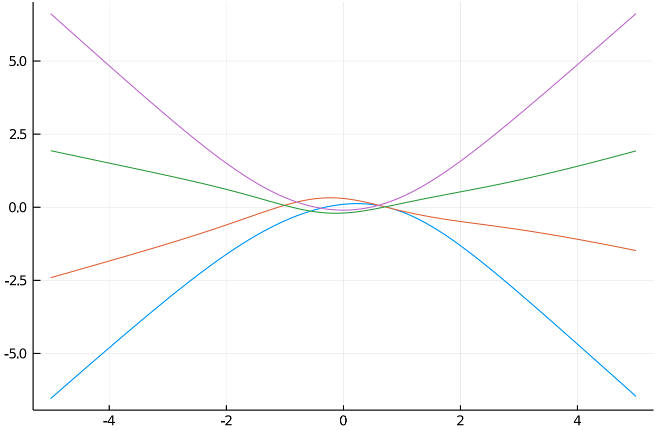
- γ_c: scaling parameter
- α_r, α_c: risk-regularity balance parameters

Toda lattice

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \\ & \vdots & & & \ddots \end{pmatrix}$$



Toda trajectories



Flaschka's change of variables

$$b_j = -\frac{1}{2} p_j$$

$$a_{jk} = \frac{1}{2} e^{\frac{1}{2}(q_j - q_k)} \text{ if } k = j + 1$$

$$A = \begin{pmatrix} 0 & a_{12} & 0 & 0 & \\ 0 & 0 & a_{23} & 0 & \cdots \\ 0 & 0 & 0 & a_{34} & \\ & \vdots & & & \ddots \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & 0 & 0 & \\ 0 & b_2 & 0 & \cdots \\ 0 & 0 & b_3 & \\ & \vdots & & \ddots \end{pmatrix}$$

Lax pair and isospectral flow

- $L = A + A^T + B$
- $M = A - A^T$

$$\dot{L} = [M, L] = ML - LM$$

Lax pair and isospectral flow

Eigenvalues of L conserved

$$\begin{aligned} H &= \frac{1}{2}\operatorname{tr} L^2 \\ &= \frac{1}{2}\sum_j b_j^2 + \sum_{j,k} w_{jk} a_{jk}^2 \\ &\propto \frac{1}{2}\sum_j p_j^2 + \sum_{j,k} w_{jk} e^{q_j - q_k} \end{aligned}$$

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \\ & \vdots & & & \ddots \end{pmatrix}$$

Generalized W

$$A = \begin{pmatrix} 0 & \sqrt{w_{12}}a_{12} - i\sqrt{w_{21}}a_{21} & \sqrt{w_{13}}a_{13} - i\sqrt{w_{31}}a_{31} & \cdots \\ 0 & 0 & \sqrt{w_{23}}a_{23} - i\sqrt{w_{32}}a_{32} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$B = \begin{pmatrix} b_1 & 0 & \cdots \\ 0 & b_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$M = A - A^*$$

$$L = A + A^* + B$$

Generalized W

Why that particular A and B?

- Give up on p's and q's

Generalized W

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- Give up on p's and q's
- $X_2 = \frac{1}{2} \operatorname{tr} L^2$ is still conserved

$$X_2 = \frac{1}{2} \sum_j b_j^2 + \sum_{j,k} w_{jk} a_{jk}^2$$

Generalized W

Why that particular A and B?

- Give up on p's and q's
- $X_2 = \frac{1}{2} \operatorname{tr} L^2$ is still conserved
- Dynamics independent of renumbering

Generalized W

Why that particular A and B?

- Give up on p's and q's
- $X_2 = \frac{1}{2} \operatorname{tr} L^2$ is still conserved
- Dynamics independent of renumbering
- All $a_{jk} \to 0$ as $t \to \infty$
- b_j converges to j-th eigenvalue
- No apparent ordering information

Back to real particle dynamics

$$R(q; W, \gamma_{\mathrm{r}}) = \sum_{j,k} w_{jk} e^{\gamma_{\mathrm{r}} \cdot (q_j - q_k)}$$

$$C(q; \gamma_{\mathrm{c}}) = \sum_{j} \cosh\left(\gamma_{\mathrm{c}} q_j\right)$$

$$H = \frac{1}{2} \sum_{j} p_j^2 + \alpha_{\mathrm{r}} R(q) + \alpha_{\mathrm{c}} C(q)$$

Rankings from trajectories

- Estimate trajectory $q(t), p(t)$ for $t \in [0, T]$
- Choose position vector q

  Permutation τ: sort elements of q

Rankings from trajectories

- Estimate trajectory $q(t), p(t)$ for $t \in [0, T]$
- Choose position vector q
  Permutation τ: sort elements of q
- *Trajectory minimum ranking potential:*

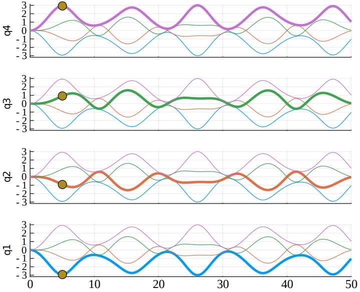
$$q^{\mathrm{TMR}} = \operatorname{minimize} R(q(t))$$

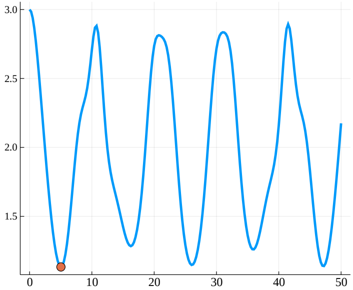
- *Average integral:*

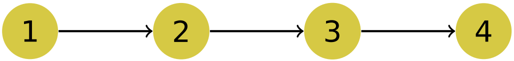
$$q^{\mathrm{AI}} = \frac{1}{T} \int_0^T q(t)\, \mathrm{d}t$$

Low-dimensional examples

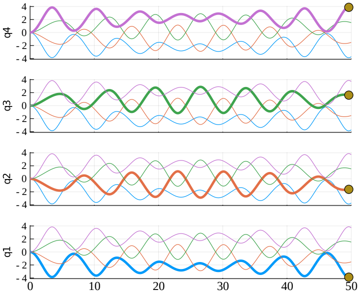
Chain of four

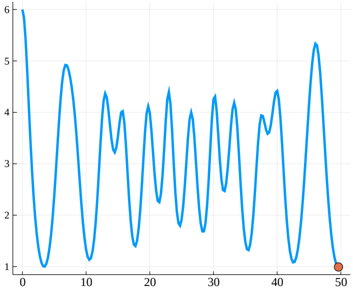


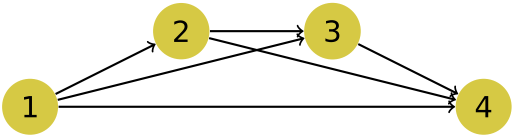




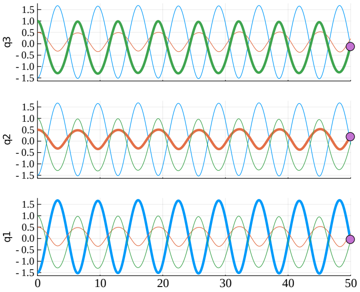
Complete tournament

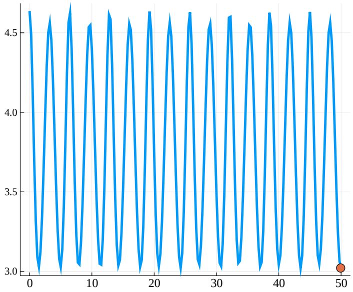


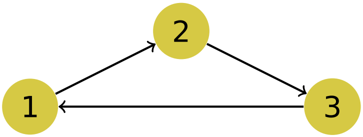




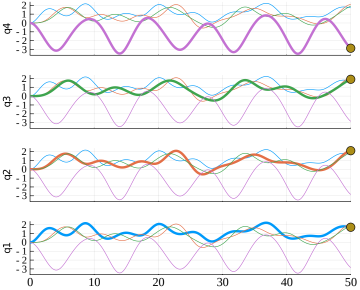
Cycle

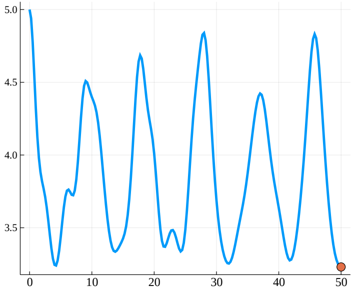

q3
q2
q1
1.5
1.0
0.5
0.0
- 0.5
- 1.0
- 1.5
0
10
20
30
40
50

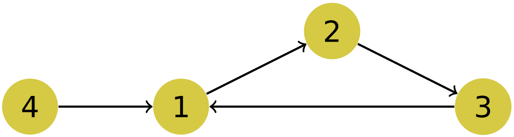

4.5
4.0
3.5
3.0
0
10
20
30
40
50


2
1
3

Cycle and one more







Synthetic data

- Define a strength s_j for each item x_j
- Probability $x_j \prec x_k$:

$$P(j,k) = \frac{1}{1 + e^{\beta(s_j - s_k)}}$$

- Low $\beta \iff$ more upsets

Synthetic data

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- Probability $x_j \prec x_k$:

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- Low $\beta \iff$ more upsets
- Linear: $s = (1, 2, 3, \ldots)$
- Two groups: $s = (0, 0, \ldots, 1, 1, \ldots)$
- Unordered: $s = (0, 0, \ldots)$

Evaluating an ordering

- Ranking potential: $R(q)$
- Tournament score:

$$\mathrm{TS}(\tau; W) = \sum_{\tau(j)<\tau(k)} w_{jk}$$

- Inversion count:

  Assuming correct order is $x_1, x_2, \ldots$

  How many swaps to bubble sort τ?

Ranking potential: Linear strengths, $\beta = 0.5$

Lower R should be better

Paired t-tests, 95% confidence intervals:

- Traj min R minus Avg intg: $(-13.9678, -13.425)$

Strongest to weakest:

Trajectory minimum R, Average integral

Tournament scores: Linear strengths, $\beta = 0.5$

High tournament score means better

Paired t-tests, 95% confidence intervals:

- Traj min R minus Avg intg: $(-0.5882, -0.4358)$
- Traj min R minus RankBoost: $(0.4528, 0.6212)$
- Avg intg minus RankBoost: $(0.9574, 1.1406)$

Strongest to weakest:

Average integral, Trajectory minimum R, RankBoost

Inversion counts: Linear strengths, $\beta = 0.5$

Low inversion count means better

Paired t-tests, 95% confidence intervals:

- Traj min R minus Avg intg: $(-0.1749, -0.0051)$
- Traj min R minus RankBoost: $(-0.519, -0.347)$
- Avg intg minus RankBoost: $(-0.4394, -0.2466)$

Strongest to weakest:

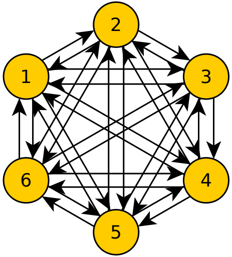
Trajectory minimum R, Average integral, RankBoost

But. . .

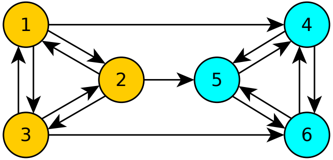
- When $\beta = 0.25$, Trajectory min R achieves higher TS and lower IC than Average integral
- When $\beta = 1.0$, RankBoost achieves lower IC than Average integral

Overall the three algorithms are roughly comparable

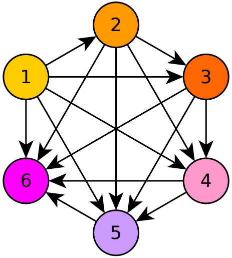
Rankability



Unordered

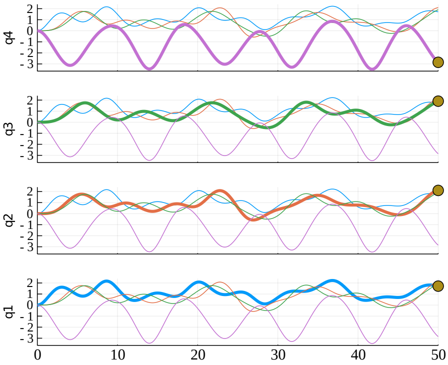


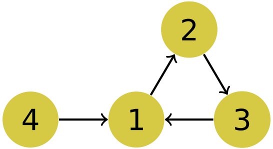
Two groups



Linear

Rankability measured by spread





Spread: $q_N - q_1$

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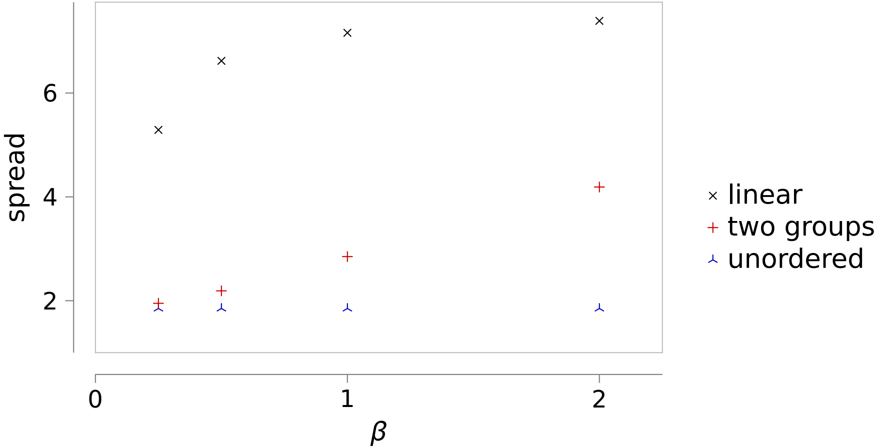
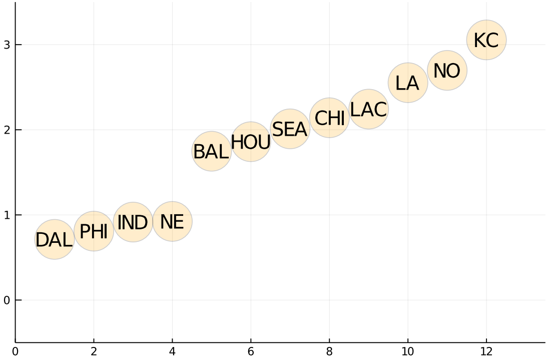
Spread for Average integral



NFL season, 2018-19

Use Average integral, rate teams based on regular season point spreads:



NFL season, 2018-19, playoffs

	j	k	$x_j - x_k$	TMR	AI	RB
1	HOU	IND	−14	.025	.142	.135
2	DAL	SEA	2	−.201	−.195	−.198
3	BAL	LAC	−6	.008	−.074	−.058
4	CHI	PHI	−1	.336	.201	.204
5	KC	IND	18	.177	.322	.322
6	LA	DAL	8	.344	.277	.256
7	NE	LAC	13	−.011	−.199	−.180
8	NO	PHI	6	.369	.284	.285
9	NO	LA	−3	.046	.022	.041
10	KC	NE	−6	.144	.321	.332
11	LA	NE	−10	.124	.245	.224

Conclusion

- Hamiltonian dynamics with ranking potential
- Connected to the Toda lattice
- Rank items using low R particle configurations (TMR)
- Rank items using average positions (AI)
- Results comparable to RankBoost
- Spread yields confidence, rankability

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Article to appear in Physica D

https://doi.org/10.1016/j.physd.2020.132676

https://authors.elsevier.com/a/1bc2pc2Eea5GM