In this project, you will derive a formula for the solution to a non-homogeneous system of differential equations

\[ x' = Ax + f(t), x(t_0) = x_0. \]  

(1)

where \( A \) is a constant matrix. From there, you will derive a formula for variation of parameters for single-variable differential equations of arbitrary order.

\textbf{Part I.}

Step 1. Prove that \( e^{At}A = Ae^{At}. \)

Step 2. Rearrange the equation to:

\[ x' - Ax = f(t). \]  

(2)

Multiply through by \( e^{-At} \) and re-write the left-hand side as the derivative of a product of a matrix and a vector.

Step 3. Using a definite integral, remove the derivative and incorporate the initial condition.

Step 4. Solve for \( x(t). \)
Part II.

Consider the $n$-th order differential equation
\[ L[y] = f(t), \] where \( L[y] = \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y. \) (3)
(The coefficients \( a_j \) are constants.)

Let \( v \) be the solution of the related homogeneous equation \( L[y] = 0 \) that satisfies the initial condition \( y(0) = 0, y'(0) = 0, \ldots, y^{(n-2)}(0) = 0, y^{(n-1)}(0) = 1. \)

Step 1. Convert equation (3) into a system of $n$ first-order equations \( x' = Ax. \)

Step 2. Show that
\[
\begin{pmatrix}
  v(t) \\
  v'(t) \\
  \vdots \\
  v^{(n-1)}(t)
\end{pmatrix}
\] is the $n$-th column of \( e^{At}. \)

Step 3. Show that
\[
y(t) = \int_0^t v(t - s)f(s)ds \quad (4)
\]
is the solution of (3) that satisfies the initial condition \( y(0) = 0, y'(0) = 0, \ldots, y^{(n-1)}(0) = 0. \)

Thus, we have found a particular solution from the complementary function.