Variation of parameters

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March 26, 2004

In this project, you will derive a formula for the solution to a non-homogeneous system of differential equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t), \mathbf{x}(t_0) = \mathbf{x}_0. \tag{1}$$

where $\mathbf{A}$ is a constant matrix. From there, you will derive a formula for variation of parameters for single-variable differential equations of arbitrary order.

Part I.

Step 1. Prove that $e^{\mathbf{A}t}\mathbf{A} = \mathbf{A}e^{\mathbf{A}t}$.

Step 2. Rearrange the equation to:

$$\mathbf{x}' - \mathbf{A}\mathbf{x} = \mathbf{f}(t). \tag{2}$$

Multiply through by $e^{-\mathbf{A}t}$ and re-write the left-hand side as the derivative of a product of a matrix and a vector.

Step 3. Using a definite integral, remove the derivative and incorporate the initial condition.

Step 4. Solve for $\mathbf{x}(t)$.

Part II.

Consider the n-th order differential equation

$$L[y] = f(t), \text{ where } L[y] = \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y. \tag{3}$$

(The coefficients a_j are constants.)

Let v be the solution of the related homogeneous equation $L[y] = 0$ that satisfies the initial condition $y(0) = 0, y'(0) = 0, \ldots, y^{(n-2)}(0) = 0, y^{(n-1)}(0) = 1$.

Step 1. Convert equation (3) into a system of n first-order equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Step 2. Show that

$$\begin{pmatrix} v(t) \\ v'(t) \\ \vdots \\ v^{(n-1)}(t) \end{pmatrix}$$

is the n-th column of $e^{\mathbf{A}t}$.

Step 3. Show that

$$y(t) = \int_0^t v(t-s) f(s) ds \tag{4}$$

is the solution of (3) that satisfies the initial condition $y(0) = 0, y'(0) = 0, \ldots, y^{(n-1)}(0) = 0$.

Thus, we have found a particular solution from the complementary function.