MATH 131 COMPUTER LAB 1: SLOPE FIELDS AND SOLUTION CURVES

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Arrange to meet with your partner soon so that you can get familiar with the project and the software package and with each other. If you have trouble working with your partner, see me as soon as possible. Each pair must turn in one write-up with both of your names on it, and you will receive a joint grade (unless there are extenuating circumstances). You may use Maple, Matlab, or Mathematica, or any similar program that allows you to do everything requested in this lab. If you are new to computer algebra systems, I recommend Maple for starters as it has the easiest learning curve.

Below, you will see instructions for a number of "deliverables." To get full credit, you must turn in all plots requested under each deliverable, write up all requested calculations, and write a paragraph answering each question. Please type your write-up. You may write in equations by hand if you like, but please be neat about it.

You'll be working with the differential equation

$$\frac{dy}{dx} = \sin(x - y). \tag{1}$$

Deliverable #1: Does equation (1) satisfy the hypotheses for the existence and uniqueness theorem for initial value problems?

Deliverable #2: Plot a slope field for this differential equation in your computer algebra system. Let x and y range from -10 to 10. Add a few solution curves with various initial conditions. Notice that the solutions tend to be curved at some point, but almost linear farther to the left and right of the curve.

Deliverable #3: Find all linear functions that solve the differential equation. That is, substitute $y = ax + b$ and find all values of a and b that yield a solution.

Deliverable #4: Look at the direction field in figure 1. It shows the solution with the initial condition $y(0) = 0$. It was generated in Maple by the command

```
DEplot(diff(y(x),x)=sin(x-y(x)), y(x), x=-100..100,
  [[0,0]], y=-100..100, linecolor=blue);
```

How can you tell by looking at this picture that it must be wrong?[1]

Date: August, 2004.

[1] Just so you'll know, the reason Maple gets it wrong has to do with its numerical method. It draws the solution curve by starting at $(0, 0)$ and taking steps that follow the direction field. Without a hint, it takes steps that are too big, but if you add the option `stepsize=0.1` to the plot command, it takes smaller steps and draws a much better picture. See figure 2. We'll learn more about numerical methods later in the course. In general, there's no sure way to tell if the step size is too big; you have to find a way to double check that your numerical calculations are reasonable somehow.

After a lot of work, a computer or a persistent human being can construct the following general solution to equation (1):

$$y(x) = f(x) = x - 2\tan^{-1}\left(\frac{x-2-C}{x-C}\right) \tag{2}$$

Deliverable #5: Plot the general solution $f(x)$ from equation (2) where $C = 1$. Why doesn't this curve look like the curves you plotted for deliverable #2?

Deliverable #6: Find the exact symbolic solution corresponding to the initial condition $y(0) = 0$. Find the exact symbolic solution corresponding to the initial condition $y(0) = 2\pi$. Plot both of these functions on the same set of axes. (You might have to ponder a little to get this one.)

Deliverable #7: Try to find the value of C corresponding to a solution with the initial condition $y(\pi/2) = 0$. What goes wrong? Keeping in mind your response to deliverable #1, is there a solution to that initial value problem, and if so, what is it?

Deliverable #8: Let's try to shed some light on this conundrum. Find the value of C corresponding to a solution with the initial condition $y(\pi/2 - \varepsilon) = 0$ assuming that ε is a very small positive number. What happens in the limit as $\varepsilon \to 0$? How does this relate to the difficulty in deliverable #7?

Deliverable #9: What line is the solution in equation (2) asymptotic to as $x \to \infty$? What about as $x \to -\infty$? *Hints:* The following might be useful:

$$m = \lim_{x\to\infty} f'(x) = ??? \tag{3}$$

$$\lim_{x\to\infty} f(x) - mx = ??? \tag{4}$$

How does the value of C affect the asymptotes? Plot $f(x)$ and these asymptotes together on a pair of axes. How does this result compare to asymptotic behavior of the solution curves you see in the numerical solutions on the slope field?

Deliverable #10: What does this exercise tell you about the symbolic solution $f(x)$ in equation (2), as in, how much should you trust it? What about solutions drawn through numerical methods?

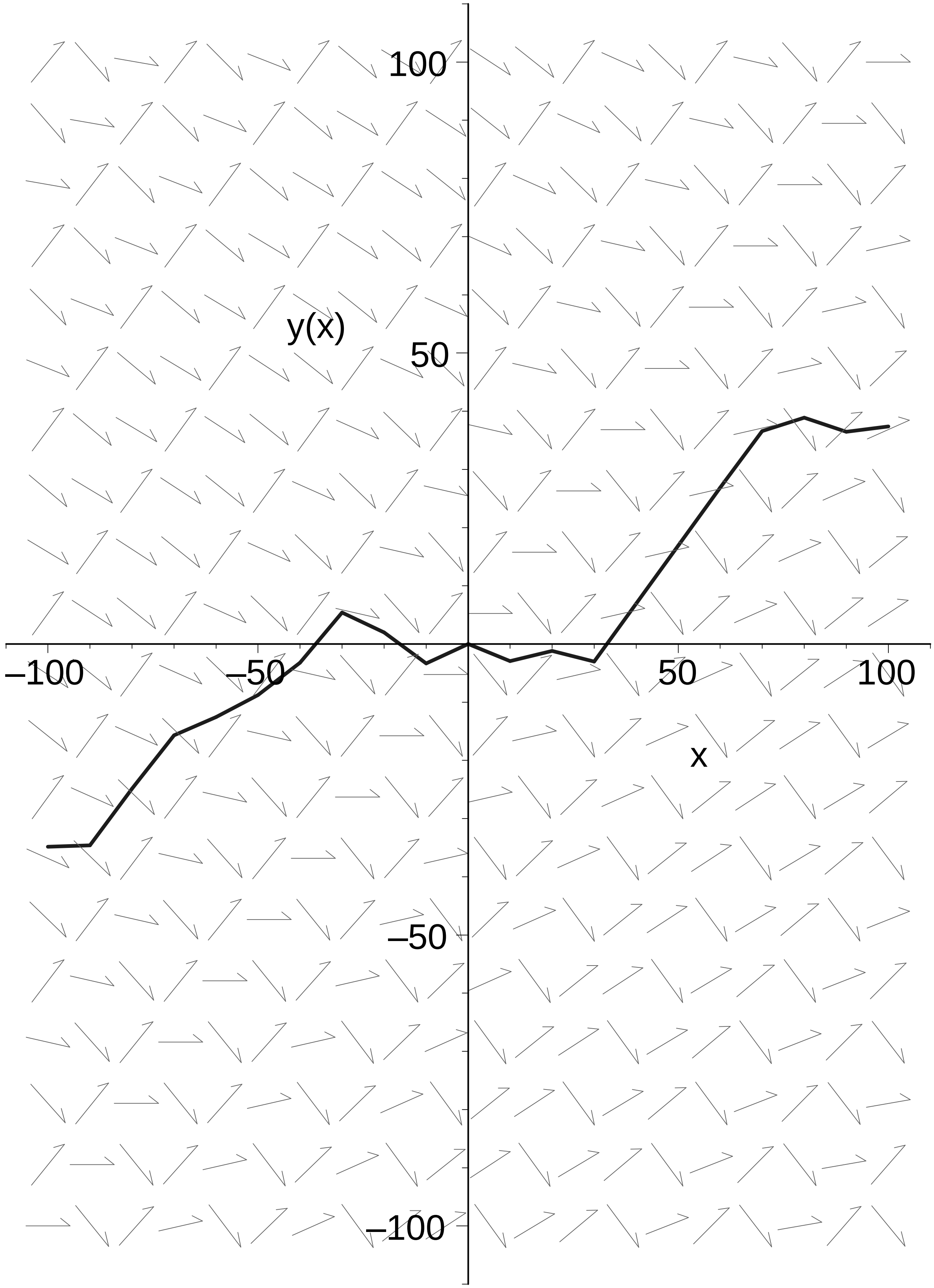

FIGURE 1. A suspicious slope field and solution where x and y range from -100 to 100.

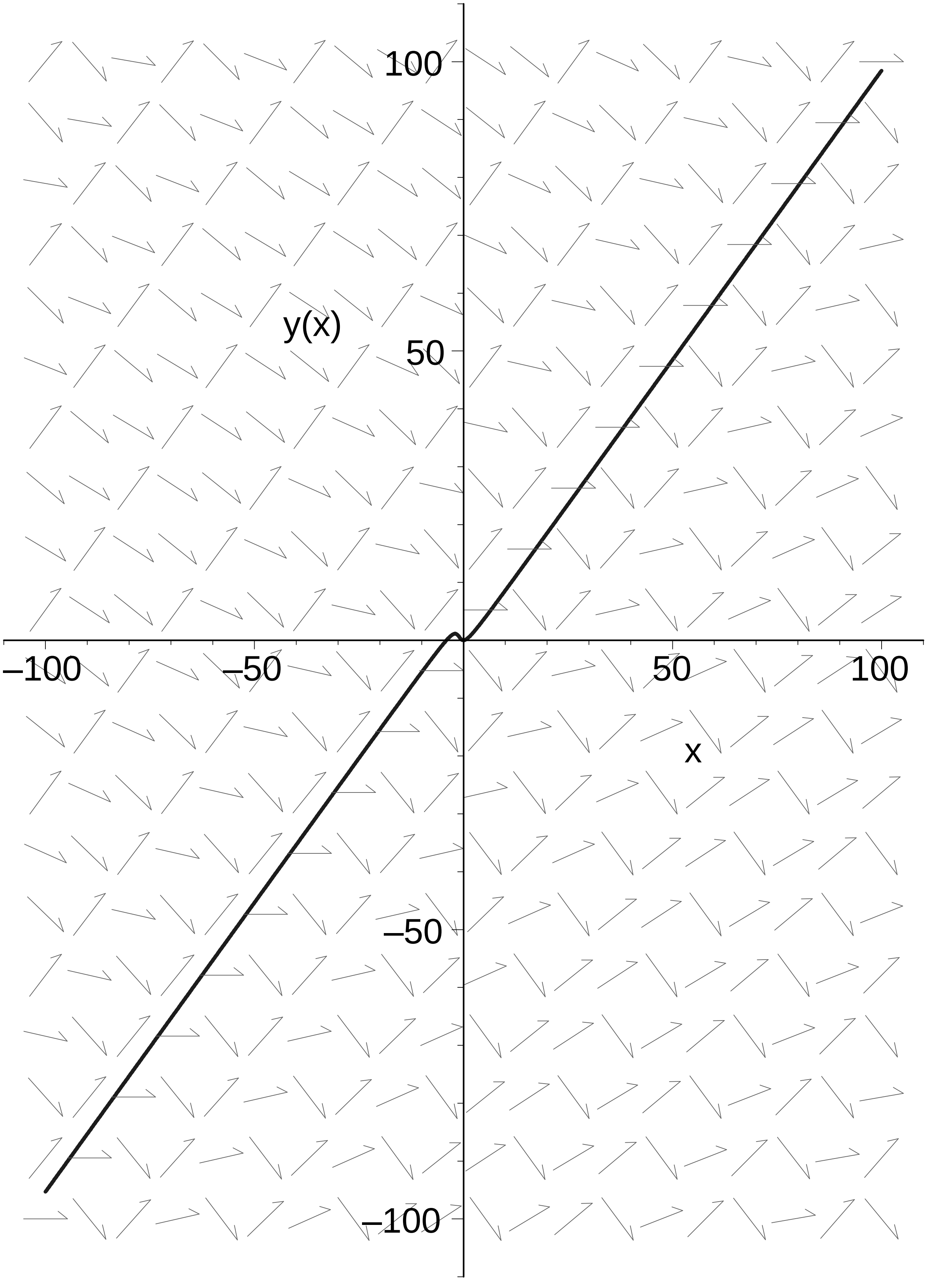

FIGURE 2. A less suspicious slope field and solution where x and y range from -100 to 100 and the step size has been reduced.