Ranking with Hamiltonian dynamics

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Ranking problems

Given

- Items $x_1, x_2, \ldots, x_n$,

- Weighted comparison matrix $W$
  
  Entry $w_{jk}$ indicates how strongly $x_j$ should be placed before $x_k$

find a permutation $\tau$ that gives a linear ordering

$$x_{\tau(1)}, x_{\tau(2)}, \ldots$$

that is as consistent as possible with $W$
Solution methods

- Brute force
- Heuristics
- Integer program
- RankBoost
- Dynamical systems
Goals

- Develop theory and practical algorithms for ranking
- Develop notion of *rankability*
- Develop interesting dynamical systems
Ranking potential

\[ R(q; W, \gamma_r) = \sum_{j,k} w_{jk} e^{\gamma_r \cdot (q_j - q_k)} \]

- \( q_j \): position of particle \( j \)
- \( \gamma_r \): scaling parameter

Seek system states such that \( R(q) \) is low
Ranking potential

\[ R(q; W, \gamma_r) = \sum_{j,k} w_{jk} e^{\gamma_r (q_j - q_k)} \]

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- \( \gamma_r \): scaling parameter

Seek system states such that \( R(q) \) is low
Hamiltonian dynamics

Confinement potential

\[ C(q; \gamma_c) = \sum_j \cosh(\gamma_c q_j) \]

Hamiltonian

\[ H = \frac{1}{2} \sum_j p_j^2 + \alpha_r R(q) + \alpha_c C(q) \]

- \( \gamma_c \): scaling parameter
- \( \alpha_r, \alpha_c \): risk-regularity balance parameters
Toda lattice

\[ W = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\vdots & \vdots & & \ddots
\end{pmatrix} \]

1 → 2 → 3 → …
Toda trajectories
Flaschka’s change of variables

\[ b_j = -\frac{1}{2} p_j \]
\[ a_{jk} = \frac{1}{2} \exp \left( \frac{1}{2} (q_j - q_k) \right) \text{ if } k = j + 1 \]

\[
A = \begin{pmatrix}
0 & a_{12} & 0 & 0 \\
0 & 0 & a_{23} & 0 \\
0 & 0 & 0 & a_{34} \\
\vdots & & & \ddots
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
b_1 & 0 & 0 \\
0 & b_2 & 0 \\
0 & 0 & b_3 \\
\vdots & & \ddots
\end{pmatrix}
\]
Lax pair and isospectral flow

- $L = A + A^T + B$
- $M = A - A^T$

$$\dot{L} = [M, L] = ML - LM$$
Lax pair and isospectral flow

Eigenvalues of $L$ conserved

\[
H = \frac{1}{2} \text{tr} L^2 = \sum_j b_j^2 + \sum_{j,k} w_{jk} a_{jk}^2
\]

\[
\propto \sum_j p_j^2 + \sum_{j,k} w_{jk} e^{q_j - q_k}
\]
Generalized $W$

\[
A = \begin{pmatrix}
0 & \sqrt{w_{12}} a_{12} - i \sqrt{w_{21}} a_{21} & \sqrt{w_{13}} a_{13} - i \sqrt{w_{31}} a_{31} & \cdots \\
0 & 0 & \sqrt{w_{23}} a_{23} - i \sqrt{w_{32}} a_{32} & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
 b_1 & 0 & \cdots \\
0 & b_2 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

\[M = A - A^*\]
\[L = A + A^* + B\]
Generalized $W$

Why that particular $A$ and $B$?

- Give up on $p$’s and $q$’s
Generalized $W$

Why that particular $A$ and $B$?

- Give up on $p$’s and $q$’s
- $X_2 = \frac{1}{2} \text{tr} \ L^2$ is still conserved

$$X_2 = \frac{1}{2} \sum_j b_j^2 + \sum_{j,k} w_{jk} a_{jk}^2$$
Generalized $W$

Why that particular $A$ and $B$?

- Give up on $p$’s and $q$’s
- $X_2 = \frac{1}{2} \text{tr } L^2$ is still conserved
- Dynamics independent of renumbering
Generalized $W$

Why that particular $A$ and $B$?

- Give up on $p$'s and $q$'s
- $X_2 = \frac{1}{2} \text{tr } L^2$ is still conserved
- Dynamics independent of renumbering
- All $a_{jk} \rightarrow 0$ as $t \rightarrow \infty$
- $b_j$ converges to $j$-th eigenvalue
- No apparent ordering information
Back to real particle dynamics

\[ R(q; W, \gamma_r) = \sum_{j,k} w_{jk} e^{\gamma_r (q_j - q_k)} \]

\[ C(q; \gamma_c) = \sum_j \cosh (\gamma_c q_j) \]

\[ H = \frac{1}{2} \sum_j p_j^2 + \alpha_r R(q) + \alpha_c C(q) \]
Rankings from trajectories

- Estimate trajectory $q(t), p(t)$ for $t \in [0, T]$
- Choose position vector $q$
  Permutation $\tau$: sort elements of $q$
Rankings from trajectories

- Estimate trajectory $q(t), p(t)$ for $t \in [0, T]$

- Choose position vector $q$
  - Permutation $\tau$: sort elements of $q$

- Trajectory minimum ranking potential:
  \[
  q^{\text{TMR}} = \text{minimize } R(q(t))
  \]

- Average integral:
  \[
  q^{\text{AI}} = \frac{1}{T} \int_{0}^{T} q(t) \, dt
  \]
Low-dimensional examples
Complete tournament

[Diagram of a complete tournament with nodes 1, 2, 3, and 4 connected in a cycle]

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Cycle

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Cycle and one more
Synthetic data

- Define a strength $s_j$ for each item $x_j$
- Probability $x_j < x_k$:
  
  $$P(j, k) = \frac{1}{1 + e^{\beta(s_j - s_k)}}$$

- Low $\beta \iff$ more upsets
Synthetic data

- Define a strength $s_j$ for each item $x_j$

- Probability $x_j < x_k$:

$$P(j, k) = \frac{1}{1 + e^{\beta(s_j - s_k)}}$$

- Low $\beta \iff$ more upsets

- Linear: $s = (1, 2, 3, \ldots)$

- Two groups: $s = (0, 0, \ldots, 1, 1, \ldots)$

- Unordered: $s = (0, 0, \ldots)$
Evaluating an ordering

- Ranking potential: $R(q)
- Tournament score:
  $$TS(\tau; W) = \sum_{\tau(j) < \tau(k)} w_{jk}$$
- Inversion count:
  Assuming correct order is $x_1, x_2, \ldots$
  How many swaps to bubble sort $\tau$?

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Ranking potential: Linear strengths, $\beta = 0.5$

Lower $R$ should be better

Paired $t$-tests, 95% confidence intervals:


Strongest to weakest:

Trajectory minimum $R$, Average integral
Tournament scores: Linear strengths, $\beta = 0.5$

High tournament score means better

Paired $t$-tests, 95% confidence intervals:

- Traj min $R$ minus Avg intg: $(-0.5882, -0.4358)$
- Traj min $R$ minus RankBoost: $(0.4528, 0.6212)$
- Avg intg minus RankBoost: $(0.9574, 1.1406)$

Strongest to weakest:

Average integral, Trajectory minimum $R$, RankBoost
Inversion counts: Linear strengths, $\beta = 0.5$

Low inversion count means better

Paired $t$-tests, 95% confidence intervals:

- Traj min $R$ minus Avg intg: $(-0.1749, -0.0051)$
- Traj min $R$ minus RankBoost: $(-0.519, -0.347)$
- Avg intg minus RankBoost: $(-0.4394, -0.2466)$

Strongest to weakest:

Trajectory minimum $R$, Average integral, RankBoost
But...

- When $\beta = 0.25$, Trajectory min $R$ achieves higher TS and lower IC than Average integral.
- When $\beta = 1.0$, RankBoost achieves lower IC than Average integral.

Overall the three algorithms are roughly comparable.
Rankability

Unordered

Two groups

Linear
Rankability measured by spread

Spread: $q_N - q_1$
Synthetic data

- Define a strength $s_j$ for each item $x_j$
- Probability $x_j < x_k$:
  $$P(j, k) = \frac{1}{1 + e^{\beta(s_j - s_k)}}$$
- Low $\beta$ $\iff$ more upsets
- Linear: $s = (1, 2, 3, \ldots)$
- Two groups: $s = (0, 0, \ldots, 1, 1, \ldots)$
- Unordered: $s = (0, 0, \ldots)$
Spread for Average integral

- linear
- two groups
- unordered

spread

\( \beta \)
NFL season, 2018-19

Use Average integral, rate teams based on regular season point spreads:
## NFL season, 2018-19, playoffs

<table>
<thead>
<tr>
<th></th>
<th>Team 1</th>
<th>Team 2</th>
<th>( x_j - x_k )</th>
<th>TMR</th>
<th>AI</th>
<th>RB</th>
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<td>SEA</td>
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</tbody>
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Conclusion

- Hamiltonian dynamics with ranking potential
- Connected to the Toda lattice
- Rank items using low $R$ particle configurations (TMR)
- Rank items using average positions (AI)
- Results comparable to RankBoost
- Spread yields confidence, rankability
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