The exponential identity

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March 26, 2004

In this project you will show that $e^{A+B}t = e^{A+B}t$ when $A$ and $B$ commute. We will assume $AB = BA$ throughout. Other matrix products may or may not commute. You might want to look at p. 285–288 in Edwards and Penney.

Step 1. Let $X(t) = e^{(A+B)t}$. Show that $X$ satisfies the initial value problem $X' = (A + B)X$ where $X(0) = I$.

Step 2. Show that $e^{At}B = Be^{At}$.

Step 3. Show that

$$\frac{d}{dt} \left( e^{At}e^{Bt} \right) = (A + B)e^{At}e^{Bt}.$$

Step 4. Use the existence and uniqueness theorem (p. 250 of Edwards and Penney) to conclude that $e^{At}e^{Bt} = e^{(A+B)t}$. Note that you will have to extend the theorem in the book to work on systems of differential equations where the unknown function is a matrix.