The exponential identity

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In this project you will show that $e^{\mathbf{A}t+\mathbf{B}t} = e^{\mathbf{A}t}e^{\mathbf{B}t}$ when $\mathbf{A}$ and $\mathbf{B}$ commute. We will assume $\mathbf{AB} = \mathbf{BA}$ throughout. Other matrix products may or may not commute. You might want to look at p. 285–288 in Edwards and Penney.

Step 1. Let $\mathbf{X}(t) = e^{(\mathbf{A}+\mathbf{B})t}$. Show that $\mathbf{X}$ satisfies the initial value problem $\mathbf{X}' = (\mathbf{A}+\mathbf{B})\mathbf{X}$ where $\mathbf{X}(0) = \mathbf{I}$.

Step 2. Show that $e^{\mathbf{A}t}\mathbf{B} = \mathbf{B}e^{\mathbf{A}t}$.

Step 3. Show that

$$\frac{d}{dt}\left(e^{\mathbf{A}t}e^{\mathbf{B}t}\right) = (\mathbf{A}+\mathbf{B})e^{\mathbf{A}t}e^{\mathbf{B}t}.$$

Step 4. Use the existence and uniqueness theorem (p. 250 of Edwards and Penney) to conclude that $e^{\mathbf{A}t}e^{\mathbf{B}t} = e^{(\mathbf{A}+\mathbf{B})t}$. Note that you will have to extend the theorem in the book to work on systems of differential equations where the unknown function is a matrix.