

# The exponential identity

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March 26, 2004

In this project you will show that  $e^{\mathbf{A}t+\mathbf{B}t} = e^{\mathbf{A}t}e^{\mathbf{B}t}$  when  $\mathbf{A}$  and  $\mathbf{B}$  commute. We will assume  $\mathbf{AB} = \mathbf{BA}$  throughout. Other matrix products may or may not commute. You might want to look at p. 285–288 in Edwards and Penney.

Step 1. Let  $\mathbf{X}(t) = e^{(\mathbf{A}+\mathbf{B})t}$ . Show that  $\mathbf{X}$  satisfies the initial value problem  $\mathbf{X}' = (\mathbf{A} + \mathbf{B})\mathbf{X}$  where  $\mathbf{X}(0) = \mathbf{I}$ .

Step 2. Show that  $e^{\mathbf{A}t}\mathbf{B} = \mathbf{B}e^{\mathbf{A}t}$ .

Step 3. Show that

$$\frac{d}{dt} (e^{\mathbf{A}t}e^{\mathbf{B}t}) = (\mathbf{A} + \mathbf{B})e^{\mathbf{A}t}e^{\mathbf{B}t}.$$

Step 4. Use the existence and uniqueness theorem (p. 250 of Edwards and Penney) to conclude that  $e^{\mathbf{A}t}e^{\mathbf{B}t} = e^{(\mathbf{A}+\mathbf{B})t}$ . Note that you will have to extend the theorem in the book to work on systems of differential equations where the unknown function is a matrix.